On the category of pre-Calabi-Yau algebras

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Context

Pre-Calabi-Yau structures on a finite dimensional graded vector space are equivalent to A_∞-structures on a certain graded vector space with additional properties. A natural question is about the link between A_∞-morphisms and pre-Calabi-Yau morphisms.
Classical examples of pre-Calabi-Yau structures on graded vector spaces are double Poisson structures on dg algebras ([Yeu18]).
Moreover, there is a result relating morphisms of double Poisson dg algebras and strict A_∞-morphisms (see Theorem 1).

Theorem 1 [FH21]

Given finite dimensional double Poisson dg algebras A, B and a morphism $\Phi : A \to B$ of double Poisson dg algebras, there exists cyclic A_∞-structures on $A \oplus A^*[d-1]$ and $B \oplus B^*[d-1]$ and a cyclic A_∞-structure on $A \oplus B^*[d-1]$ together with strict cyclic A_∞-morphisms $A[1] \oplus B^*[d] \to A[1] \oplus A^*[d]$ and $A[1] \oplus B^*[d] \to B[1] \oplus B^*[d]$.

Settings

- $\Bbbk$: field of characteristic $\neq 2, 3$
- $d \in \mathbb{Z}$
- A, B : graded vector spaces
- $\mathcal{H}om_\Bbbk(A, B)$: graded vector space formed by sums of linear homogeneous maps $A \to B$
- $A^* = \mathcal{H}om_\Bbbk(A, \Bbbk)$
- $\mathcal{T} = \bigsqcup_{k>0} \mathcal{T}_k$ with $\mathcal{T}_k = \mathbb{N}^k$ for $k > 1$ and $\mathcal{T}_1 = \mathbb{N}^*$
- C_n: cyclic group with n elements
- $\tau_{V_1,\dots,V_n} : V_1 \otimes \cdots \otimes V_n \to V_n \otimes V_1 \otimes \cdots \otimes V_{n-1}$ given by $\tau_{V_1,\dots,V_n}(v_1 \otimes \cdots \otimes v_n) = (-1)^\epsilon (v_n \otimes v_1 \otimes \cdots \otimes v_{n-1})$ where V_i are graded vector spaces, $v_i \in V_i$ and

$$\epsilon = |v_n| \sum_{i=1}^{n-1} |v_i|$$

Multidual

The **multidual of A and B** is the graded vector space $\mathrm{Multi}_d(A, B)$ defined by

$$\mathrm{Multi}_d(A, B) = \prod_{k \in \mathbb{N}^*} \prod_{(n_1,\dots,n_k) \in \mathcal{T}_k} \mathrm{Multi}_d^{n_1,\dots,n_k}(A, B)$$

$\mathrm{Multi}_d^{n_1,\dots,n_k}(A, B) = \mathcal{H}om_\Bbbk(A[1]^{\otimes n_1} \otimes \cdots \otimes A[1]^{\otimes n_k}, B[-d]^{\otimes k})$.
We will denote $\mathrm{Multi}_d(A, A)$ simply by $\mathrm{Multi}_d(A)$.

The space of invariants

The **action of $\prod_{n \in \mathbb{N}^*} C_n$ on $\mathrm{Multi}_d(A, B)$** is given by the element $(\sigma \cdot F) \in \mathrm{Multi}_d(A, B)$ defined as

$$(\sigma \cdot F)^{n_1,\dots,n_k} = \tau_{B[-d],\dots,B[-d]} \circ F^{n_k,n_1,\dots,n_{k-1}} \circ \tau_{A[1]^{\otimes n_1},\dots,A[1]^{\otimes n_k}}$$

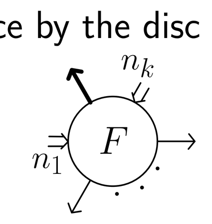
for $(n_1, \dots, n_k) \in \mathcal{T}_k$ and $\sigma = (1 \dots k)$.
We denote $\mathrm{Multi}_d(A, B)^{C_\bullet}$ the space of elements of $\mathrm{Multi}_d(A, B)$ that are invariant under the action of $\prod_{n \in \mathbb{N}^*} C_n$.

Diagrammatic representation

We have that

$$\mathrm{Multi}_d^{n_1,\dots,n_k}(A, B)[d+1] \simeq \mathcal{H}om_\Bbbk(A[1]^{\otimes n_1} \otimes \cdots \otimes A[1]^{\otimes n_k}, B[-d]^{\otimes (k-1)} \otimes B[1])$$

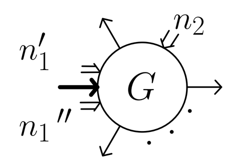
We represent an element F in this space by the disc



where the bold arrow indicates the shift of the output and the double arrows represent n_i consecutive arrows. We also have for $n_1' + 1 + n_1'' = n_1$ that

$$\mathrm{Multi}_d^{n_1,\dots,n_k}(A, B)[d+1] \simeq \mathcal{H}om_\Bbbk(A[1]^{\otimes n_1'} \otimes A[-d] \otimes A[1]^{\otimes n_1''} \otimes \cdots \otimes A[1]^{\otimes n_k}, B[-d]^{\otimes k})$$

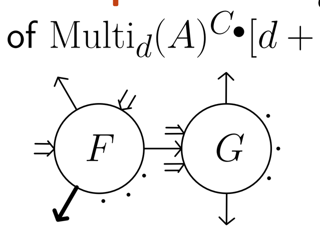
We represent an element G in this space by the disc



We will use those two representations for elements of $\mathrm{Multi}_d(A, B)^{C_\bullet}[d+1]$ and compose the discs by connecting the bold arrow of one with an arrow which is not bold.

Necklace product

The **necklace product** of $s_{d+1}F, s_{d+1}G \in \mathrm{Multi}_d(A)^{C_\bullet}[d+1]$ is the element of $\mathrm{Multi}_d(A)^{C_\bullet}[d+1]$ represented by the sum of



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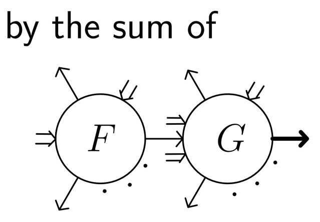


Pre-Calabi-Yau algebras

A **d-pre-Calabi-Yau structure on A** is a Maurer-Cartan element of the graded Lie algebra $(\mathrm{Multi}_d(A)^{C_\bullet}[d+1], [-,-]_{\mathrm{nec}})$ where $[-,-]_{\mathrm{nec}}$ is the **necklace bracket** given as the graded commutator of the necklace product, *i.e.*

$$[s_{d+1}F, s_{d+1}G]_{\mathrm{nec}} = s_{d+1}F \underset{\mathrm{nec}}{\circ} s_{d+1}G - (-1)^{(|F|+d+1)(|G|+d+1)} s_{d+1}G \underset{\mathrm{nec}}{\circ} s_{d+1}F$$

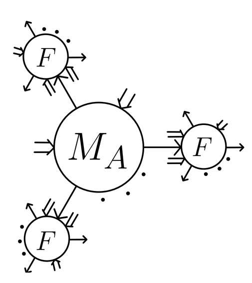
A endowed with a d-pre-Calabi-Yau structure is called a **d-pre-Calabi-Yau algebra**.

Theorem 2 [KTV21])

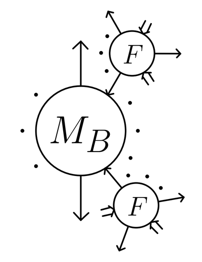
A d-pre-Calabi-Yau structure on A induces a cyclic A_∞-structure on $A \oplus A^*[d-1]$ such that $A \hookrightarrow A \oplus A^*[d-1]$ is an A_∞-subalgebra. Moreover, those data are equivalent if A is finite dimensional.

The category pCY_d [KTV21, LV22]

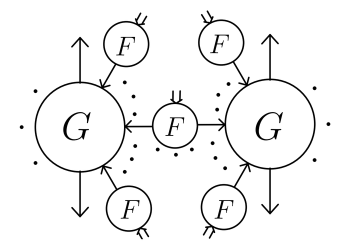
- **objects** : pre-Calabi-Yau algebras
- **morphisms** : elements $s_{d+1}F \in \mathrm{Multi}_d(A, B)^{C_\bullet}[d+1]$ of degree 0 such that the elements represented by the following diagrams are equal:



and

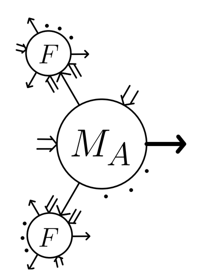


- **composition** : given elements $s_{d+1}F \in \mathrm{Multi}_d(A, B)^{C_\bullet}[d+1]$ and $s_{d+1}G \in \mathrm{Multi}_d(B, C)^{C_\bullet}[d+1]$ their composition is the element $s_{d+1}G \circ s_{d+1}F \in \mathrm{Multi}_d(A, C)^{C_\bullet}[d+1]$ represented by diagrams of the form

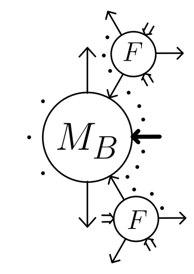


Theorem 3 [Bou23]

There is a functor $\mathcal{P} : pCY_d \to \widehat{A_\infty}$ sending a d-pre-Calabi-Yau structure on A to the cyclic A_∞-structure on $A \oplus A^*[d-1]$ given in Theorem 2 and a d-pre-Calabi-Yau morphism $s_{d+1}F \in \mathrm{Multi}_d(A, B)[d+1]^{C_\bullet}$ to the triple $(sm_{A \oplus B^*}, \varphi_A, \varphi_B)$ where $sm_{A \oplus B^*}$ is the A_∞-structure on $A \oplus B^*[d-1]$ defined by maps F_A and F_B represented by



and



respectively, φ_A is the A_∞-morphism defined by $\varphi_A^1(sa) = sa$ for $sa \in A[1]$, $\varphi_A(tf) = tf \circ s_{d+1}F^1[-d]$ for $tf \in B^*[d]$ and for $(n_1, \dots, n_k) \in \mathcal{T}_k$, $\varphi_A^{n_1,\dots,n_k}$ is the map $\bigotimes_{i=1}^{k-1} (A[1]^{n_i} \otimes B^*[d]) \otimes \mathcal{A}[1]^{n_k} \to A^*[d]$ given by

$$\begin{aligned}\varphi_A^{n_1,\dots,n_k}&(s\bar{a}^1, tf^1, \dots, s\bar{a}^{k-1}, tf^{k-1}, s\bar{a}^k)(s_{-d}b) \\ &= \pm((f^{k-1} \circ s_d) \otimes \cdots \otimes (f^1 \circ s_d))(\Phi^{n_k+1+n_1,n_{k-1},\dots,n_2}(s\bar{a}^k \otimes sb \otimes s\bar{a}^1, s\bar{a}^{k-1}, \dots, s\bar{a}^2))\end{aligned}$$

and φ_B is the A_∞-morphism given by $\varphi_B^1(sa) = s_{d+1}F^1(sa)$ for $sa \in A[1]$, $\varphi_B(tf) = tf$ for $tf \in B^*[d]$ and for $n > 1$ $\varphi_B^{n_1,\dots,n_k}$ is the map $\bigotimes_{i=1}^{k-1} (A[1]^{n_i} \otimes B^*[d]) \otimes \mathcal{A}[1]^{n_k} \to B[1]$ given by

$$\begin{aligned}\varphi_B^{n_1,\dots,n_k}&(s\bar{a}^1, tf^1, \dots, s\bar{a}^{k-1}, tf^{k-1}, s\bar{a}^k) \\ &= \pm((f^{n-1} \circ s_d) \otimes \cdots \otimes (f^1 \circ s_d) \otimes \mathrm{id})(s_{d+1}\Phi^{n_k,n_{k-1},\dots,n_1}(s\bar{a}^k, \dots, s\bar{a}^1))\end{aligned}$$

where the signs in the expressions of φ_A and φ_B are completely determined by the Koszul rules.

The partial category $\widehat{A_\infty}$

- **objects** : A_∞-algebras of the form $A \oplus A^*[d-1]$ for a graded vector space A
- **morphisms** : triple $(sm_{A \oplus B^*}, \varphi_A, \varphi_B)$ where $sm_{A \oplus B^*}$ is an A_∞-structure on $A \oplus B^*[d-1]$ and φ_A, φ_B are A_∞-morphisms

$$\begin{tikzcd} & A[1] \oplus B^*[d] \arrow[dl, "\varphi_A"'] \arrow[dr, "\varphi_B"] & \\ A[1] \oplus A^*[d] & & B[1] \oplus B^*[d] \end{tikzcd}$$

- **composition** : two morphisms $(sm_{A \oplus B^*}, \varphi_A, \varphi_B)$ and $(sm_{B \oplus C^*}, \psi_B, \psi_C)$ are composable if there exists a triple $(sm_{A \oplus C^*}, \chi_A, \chi_C)$ such that the square in the diagram

$$\begin{tikzcd} & & A[1] \oplus C^*[d] \arrow[dl, "\chi_A"'] \arrow[dr, "\chi_B"] & & \\ & A[1] \oplus B^*[d] \arrow[dl, "\varphi_A"'] \arrow[dr, "\varphi_B"] & & B[1] \oplus C^*[d] \arrow[dl, "\psi_A"'] \arrow[dr, "\psi_B"] & \\ A[1] \oplus A^*[d] & & B[1] \oplus B^*[d] & & C[1] \oplus C^*[d] \end{tikzcd}$$

commutes. In this case, the composition is the triple $(sm_{A \oplus C^*}, \phi_A \circ \chi_A, \psi_C \circ \chi_C)$.

We remark that an A_∞-structure on $A \oplus B^*[d-1]$ is tantamount to the data of maps

$$F_A : A[1]^{\otimes n_1} \otimes \cdots \otimes A[1]^{\otimes n_k} \to B[-d]^{\otimes (k-1)} \otimes A$$

and

$$F_B : A[1]^{\otimes n_k} \otimes B[-d] \otimes A[1]^{\otimes n_1} \otimes \cdots \otimes A[1]^{\otimes n_{k-1}} \to B[-d]^{\otimes (k-1)}$$

satisfying some properties.

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