GRENOBLE JT WORK w. CALAQUE - SCHEICOITZ 1

>>> FROM CY to quasi Bisymplectic Structures

context: Kontsevich - Rosenberg principle

$$A \text{ ass. } k\text{-alg } (\text{char} = 0) \overset{N \geq 0}{\rightsquigarrow} X = \underbrace{\mathrm{Rep}_k(A, N)}_{\mathrm{Hom}_{k\text{-alg}}(A, M_N(k))} / GL_N$$

alg structure $\overset{?}{\rightsquigarrow}$ geom. quality

Examples '06 '07

Bisymplectic	$\xrightarrow{[CBEG, E]}$	Hamiltonian
dble Poisson	$\xrightarrow{[VdB]}$	Poisson
qdble P	$\xrightarrow{''}$	q Poisson

$\cap$ [IKV'19]

$\cap$ [FH'20] preCY

dg-cats (rel) CY $\xrightarrow[\text{[BD'16]}]{\text{Perf} \,\supseteq\, \mathrm{Rep}/GL_N}$ (log) symp der. stacks

[KTV] CY $\longleftrightarrow$ NDpreCY

Example

M compact oriented mfld w. boundary

$\partial M = N^- \sqcup N^+$

dg nerve sing simpl set

get a CY cospan

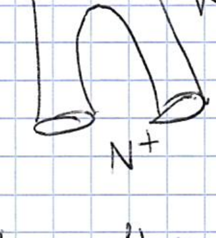
$$dgSing N^- \longrightarrow dgSing M \longleftarrow dgSing N^+$$

$\simeq_q$ SI (M conn)

$C(M)$ dga of chains on based loop sp.

modules / are local systems

Subex:



$$k[z^{\pm 1}] \xrightarrow[z = xy]{} k\langle x^{\pm 1}, y^{\pm 1}\rangle \longleftarrow k[x^{\pm 1}] \sqcup k[y^{\pm 1}]$$

Goal: CY mult moment map $\rightarrow$ q Bisymp on the LHS

- nCY-str

(2)

Def * nCY struct on A ~~smooth~~ smooth dgcat is $c: k[n] \to HC^-(A)$
~~st~~ which induces $k[n] \to HH(A)$ which induces
$A^\vee[n] \xrightarrow{\sim} A$ "ND"

* nCY struct on $B \to A \leftarrow C$ is

$$\begin{tikzcd} k[n] \arrow[r, "c_B"] \arrow[d, "c_C"'] & HC^-(B) \arrow[d] \\ HC^-(C) \arrow[r] \arrow[ur, phantom, "G"] & HC^-(A) \end{tikzcd}$$

st c_B, c_C ND &

$$\begin{tikzcd} A^\vee[n] \arrow[r] \arrow[d] & B^\vee[n] \simeq B \arrow[d] \\ C^\vee[n] \simeq C \arrow[r] & A \end{tikzcd}$$

(ho) cartesian.

Examples

* $A = k[x]$ $\underset{\text{bimod}}{\simeq} \left(k[x] \otimes kx \otimes k[x] \to k[x] \otimes k[x] \right)$
1-CY
S^1

$1 \otimes x \otimes 1 \longmapsto x \otimes 1 - 1 \otimes x$

$1 \otimes 1 \longmapsto x \otimes x^* \otimes 1 - 1 \otimes x^* \otimes x$

$1 \otimes 1 \mapsto 1 \otimes x^*$

$A^\vee[1] \simeq (k[x] \otimes k[x] \longrightarrow k[x] \otimes kx^* \otimes k[x])$

$c = 1 \otimes x = dx$ lifts.

* $A = k[x^{\pm 1}]$ $c = $ "$d\log x$" $= x^{-1} \otimes x = \dfrac{x^{-1} \otimes x - x \otimes x^{-1}}{2}$

has a ND cyclic lift $\gamma = \sum_{k \geq 0} k! \dfrac{(x^{-1} \otimes x)^{k+1} - (x \otimes x^{-1})^{k+1}}{2} u^k$

* pair of pants

$$k[x^{\pm 1}] \sqcup k[y^{\pm 1}] \longrightarrow k\langle x^{\pm 1}, y^{\pm 1} \rangle \xleftarrow{z = xy} k[z^{\pm}]$$

$$x^{-1} \otimes x - x \otimes x^{-1} + y^{-1} \otimes y - y \otimes y^{-1} - (xy)^{-1} \otimes xy + xy \otimes (xy)^{-1}$$
$$= b(y^{-1} \otimes x^{-1} \otimes xy - y \otimes y^{-1} x^{-1} \otimes x)$$

deg 0 $\to$ all lifts

§ NC-forms

$$C_n(A) = A^{\otimes n+1} \xrightarrow{\sim} \Omega^n A$$
$$a_0\otimes\cdots\otimes a_n \longmapsto a_0\,da_1\cdots da_n$$

$$DR^\bullet A = \frac{\Omega^\bullet A}{[\Omega^\bullet A, \Omega^\bullet A]} \qquad D_A = \mathrm{Der}(A, A\otimes A) = (\Omega^1 A)^\vee$$

$\delta \in D_A$ lin. dble. der $\rightsquigarrow$ i_δ gr dble der $\Omega A \to \Omega A \otimes \Omega A$
$$a \longmapsto 0, \qquad da \longmapsto \delta(a)$$

eg $i_\delta(p\,dq\,dr) = p\,\delta q' \otimes \delta q''\,dr - p\,dq\,\delta r' \otimes \delta r''$

$\rightsquigarrow$ contraction $\iota_\delta = {}^\circ i_\delta : \Omega^\bullet A \to \Omega^{\bullet-1} A$

w ${}^\circ(\alpha\otimes\beta) = \pm\beta\alpha$

example $\delta = E$ gauge elt $a \mapsto a\otimes 1 - 1\otimes a$

$\rightsquigarrow$ $\iota_E(a_0\,da_1\cdots da_n) = \sum \pm [a_l, da_{l+1}\cdots da_n\,a_0\,da_1\cdots da_{l-1}]$

fact $HC^\bullet(A) \simeq H(\Omega^\bullet A[\underset{\deg -2}{u}], b - uB) \simeq \left(\dfrac{\Omega^\bullet A[u]}{[d\Omega^\bullet A, d\Omega^\bullet A]}, \iota_E - ud\right)$

Def: $(A, \omega \in DR^2 A, \Phi \in A^\times)$ qtsymp if

(i) $d\omega = \frac{1}{6}(\phi^{-1}d\phi)^3$

(ii) $\iota_E\omega = \frac{1}{2}(\phi^{-1}d\phi + d\phi\,\phi^{-1})$

(iii) $D_A \oplus A\,d\phi\,A \longrightarrow \Omega_A$, $(\delta, \eta) \longmapsto \iota_\delta(\omega) + \eta$ surj (ND)

Thm [BCS] $\Phi: k[x^{\pm1}] \to A$ 1-CY $\Rightarrow$ A qtsymp. if A 1-sm (Ω_A proj A^e-mod) dem $(\iota_E - ud)\left(\sum_{k\geq 0} u^k \omega_{k+1}\right) = \phi(\gamma)$

$$= \sum_{k\geq 0} \frac{k!}{(2k+1)!}(-u)^k \frac{(x^{-1}dx)^{2k+1} + (dx\,x^{-1})^{2k+1}}{2}$$

"$\phi = \phi(x)$"

$\Rightarrow$ $\iota_E \omega_1 = \frac{1}{2}(\phi^{-1}d\phi + d\phi\,\phi^{-1})$ (ii)

$\iota_E\omega_2 - d\omega_1 = -\frac{1}{6}\phi(\gamma_2) \Rightarrow d\omega_1 = \frac{1}{6}(\phi^{-1}d\phi)^3 \mod [-,-]$

(i).

ND? $T = k[x^{\pm 1}]$

$$A^\vee[1] \xrightarrow{\phi^\vee} T^\vee[1]\otimes_{T^e} A^e \xrightarrow[\sim]{\gamma} T\otimes_{T^e} A^e \xrightarrow{\phi} A$$

ie short res's

$$\begin{tikzcd}
A^e \arrow[r] \arrow[d, "E"'] & A^e \arrow[r, "\frac{1}{2}(\phi^{-1}\otimes 1 + 1\otimes\phi^{-1})"] \arrow[d] & A^e \arrow[r, "d\phi"] \arrow[d] & \Omega_A \arrow[d] \\
D_A \arrow[r, "ev_\phi"'] & A^e \arrow[r, "="'] & A^e \arrow[r] & A^e
\end{tikzcd}$$

w. homotopy $D_A \to \Omega_A \Rightarrow \iota_E\omega_1 = \frac{1}{2}(\phi^{-1}d\phi + d\phi\,\phi^{-1})$

ND (CY)

$$\begin{tikzcd}
A^e \arrow[r, "\frac{1}{2}(\phi^{-1}\otimes 1 + 1\otimes\phi^{-1})"] \arrow[d, "E"'] & A^e \arrow[d, "d\phi"] \\
D_A \arrow[r, "\iota(\omega_1):\, \delta \mapsto \iota_\delta(\omega_1)"'] & \Omega_A
\end{tikzcd}$$

qiso $(\ ) \simeq (\ )$

$\Rightarrow D_A \to \Omega_A/\langle d\phi\rangle$ surj ✓