Properadic calculus applied to pre-Calabi–Yau algebras

Johan Leray

Pre-Calabi–Yau day – Institut Fourier, Grenoble – 3 november 2023

Nantes University

The goals of this talk are the following :

- compare pre-Calabi–Yau algebras with others algebraic structures which are also *properadic* ;
- using this properadic description to deduce some homotopical results.

This talk is based on a joint work with Bruno Vallette (Univ. Paris Sorbonne Paris Nord).

Contents

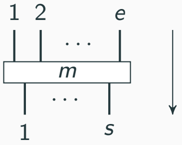
Properads and properadic gebras

$\mathfrak{S}$-bimodule

Definition

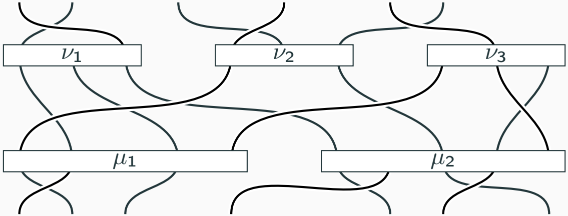
A $\mathfrak{S}$-bimodule M is a collection of dg vector spaces $\{M(s,e)\}$ indexed by $s \in \mathbb{N}^*$ and $e \in \mathbb{N}$, with a right action of $\mathfrak{S}_e$ and a left action of $\mathfrak{S}_s$.

We will represent an element m of a $\mathfrak{S}$-bimodule $M(s,e)$ by a directed graph as follows:



Connected product (Vallette)

The *connected product* $M \boxtimes N$ of two $\mathfrak{S}$-bimodules M and N is given by "the sum of directed two-levelled **connected** graphs where each vertex v with e_v inputs and s_v outputs is labelled by an element of $\nu_j \in N(s_v, e_v)$ if v is in the upper line or $\mu_j \in M(s_v, e_v)$ if v is the bottom line".



Proposition (Vallette)

The category ($\mathfrak{S}$-bimod, $\boxtimes$, I) is monoidal.

Properad (Vallette)

Definition

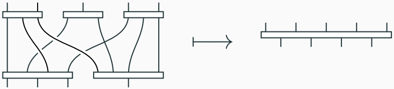
A *properad* is a monoid in the category $(\mathfrak{S}\text{-bimod}, \boxtimes, I)$.

The first important example: End_V
Let V be a dg vector space. The $\mathfrak{S}$-bimodule End_V is defined by

$$\mathrm{End}_V(s, e) = \mathrm{Hom}_k(V^{\otimes e}, V^{\otimes s})$$

and the composition of morphisms gives us the properadic structure.

$$\mathrm{End}_V \boxtimes \mathrm{End}_V \longrightarrow \mathrm{End}_V$$



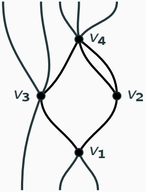
A wide class of examples:
Operads can be viewed as properads concentrated in arities $(1, *)$.

Free properad

Proposition (Vallette)

For an $\mathfrak{S}$-bimodule V, there exists a free properad $\mathscr{G}(V)$.

$$\mathscr{G}(V)(s,e) = \bigoplus$$



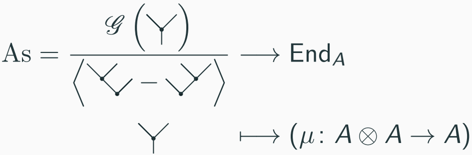
with $v_i \in V$

One can define some properads by generators and relations.

Definition

Let A be dg vector space and $\mathscr{P}$ be a properad. A *structure of $\mathscr{P}$-gebra* on A is a morphism of properads $\mathscr{P} \to \mathsf{End}_A$.

Example: An As-gebra is an associative algebra



such that $\mu(\mu \otimes A) = \mu(A \otimes \mu)$.

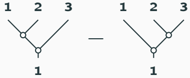
First example: DPois

Double Poisson gebras (Van den Bergh) are encoded by the properad DPois which is defined by the following presentation:

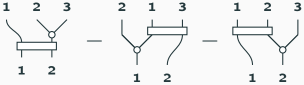
- the generators, which live in degree 0, are
  - a product in arities $(1,2)$

  - a double bracket in arities $(2,2)$

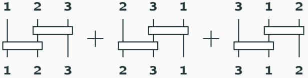
- the relations are :
  - associativity



  - derivation



  - double-Jacobi

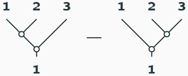


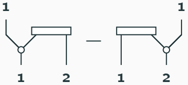
Second example: V

V gebras (Tradler–Zeinalian and Poirier–Tradler) are encoded by the properad V which is given by the following presentation:

- the generators are
  - a product in arities $(1, 2)$  in degree 0

  - a symmetric co-inner product in arities $(2, 0)$  in degree -2

- the relations are
  - associativity 

  - compatibility 

Properadic gebras up to homotopy

Motivation

With E. Hoffbeck and B. Vallette, one of our goals is to give a explicit description of the ∞-category of $\mathscr{P}$-gebras given by the localisation $\mathscr{P}\text{-geb}[(\xrightarrow{\sim})^{-1}]$ of the category of $\mathscr{P}$-gebras by quasi-isomorphisms.

We want to extend/enhance the following result available for operadic algebras

Theorem (Vallette)

Let $\mathscr{O}$ be an operad and $\mathscr{O}_\infty = \Omega(\mathscr{C})$ be a cofibrant resolution of $\mathscr{O}$. We have an equivalence of category

$$\mathrm{ho}(\mathscr{O}\text{-alg}) \simeq \mathrm{ho}(\infty - \mathscr{O}_\infty\text{-alg}).$$

where

- $\mathscr{O}$-alg *is the model category of $\mathscr{O}$-algebras with (strict) morphisms ;*
- $\infty - \mathscr{O}_\infty$-alg *is the model category where the objects are $\mathscr{O}$-algebras up to homotopy and the morphisms are ∞-morphisms.*

Motivation

Problems

Some difficulties appear for the properadic case:

- for a properad $\mathscr{P}$, we have not free $\mathscr{P}$-gebras ;
- the category $\mathscr{P}$-geb has not a model structure.

but

Some constructions are possible

- One can defined the category $\infty - \mathscr{P}_\infty$-geb with $\mathscr{P}_\infty = \Omega\mathscr{C}$.
- Some theorems can be extend from operadic case to properadic case.

Coproperad and cobar construction

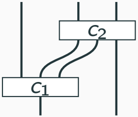
Definition

- A coproperad $(\mathscr{C}, \Delta\colon \mathscr{C} \to \mathscr{C} \boxtimes \mathscr{C})$ is a comonoid in the monoidal category $(\mathfrak{S}\text{-bimod}, \boxtimes, I)$.
- When $\mathscr{C}$ is coaugmented for $\eta : I \to \mathscr{C}$, we denote $\overline{\mathscr{C}} = \mathrm{Ker}(\eta)$.
- The cobar construction is the functor

$$\Omega\colon (\mathscr{C}, \Delta) \mapsto (\mathscr{G}(\mathrm{s}^{-1}\overline{\mathscr{C}}), \partial_\Delta)$$

where $\mathscr{G}(\overline{\mathscr{C}})$ is the (quasi-)free properad constructed on $\overline{\mathscr{C}}$ and ∂_Δ is constructed with the partial coproduct

$$\Delta_{(1,1)}\colon \overline{\mathscr{C}} \xrightarrow{\Delta} \mathscr{C} \boxtimes \mathscr{C} \xrightarrow{\pi_{(1,1)}} \mathscr{C} \boxtimes_{(1,1)} \mathscr{C}.$$



$\in (\mathscr{C} \boxtimes_{(1,1)} \mathscr{C})(2,3)$

Find cofibrant resolution via Koszul duality

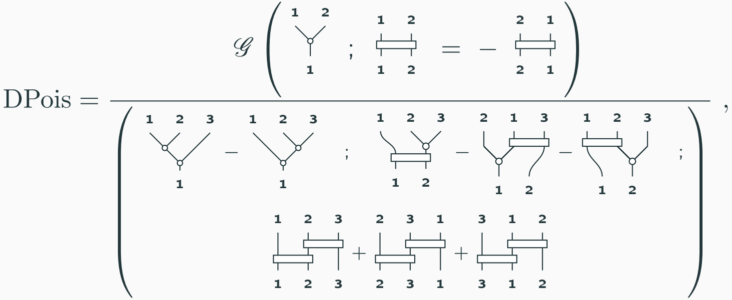
Some quadratic properads have a nice homological property, the Koszulness, which give us an explicit and minimal cofibrant resolution of it.

Definition–Proposition (Vallette)

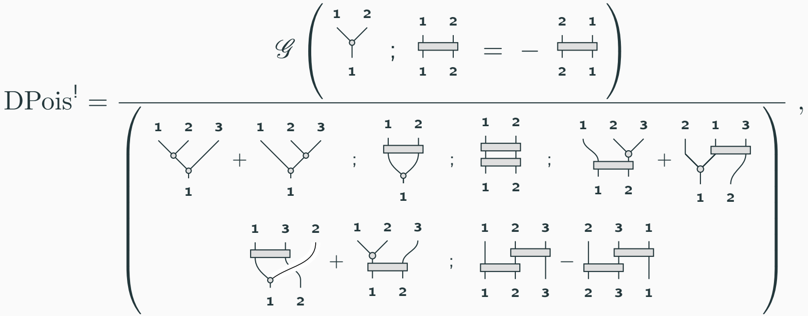
Let $\mathscr{P} = \mathscr{G}(V)/\langle R\rangle$ be a quadratic properad where V is finite dimensional.

- The Koszul dual of $\mathscr{P}$ is the coproperad
$$\mathscr{P}^{\mathrm{i}} = \left(\mathscr{P}^{!}\right)^{*} \text{ with } \mathscr{P}^{!} = \mathscr{G}(\mathrm{s}^{-1}V^{*})/\langle \mathrm{s}^{-2}R^{\perp}\rangle.$$
- If $\mathscr{P}$ is Koszul, then
$$\Omega(\mathscr{P}^{\mathrm{i}}) \xrightarrow{\sim} \mathscr{P}$$
is the minimal cofibrant resolution. In this case, the $\mathscr{P}$-gebra structure up to homotopy is encoded by $\mathscr{P}_{\infty} = \Omega(\mathscr{P}^{\mathrm{i}})$.

Recall that



then we have

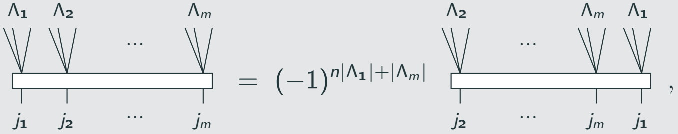


where $\nu =$  has degree -1 and  has degree -1.

The example of DPois 3

Theorem (L.)

The properad DPois *is Koszul and the properad* DPois_∞ *is generated by*



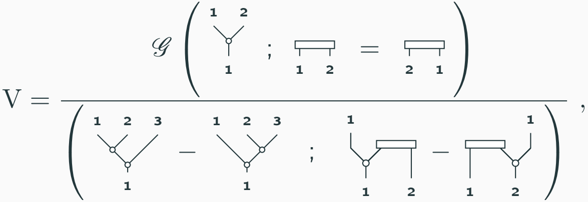
in degree $1-n$ *and where* $\Lambda_1 \sqcup \cdots \sqcup \Lambda_m = \{1, \ldots, n\}$ *is a partition of* n *(so* $|\Lambda_i| > 0$*) and* $\{j_1, \ldots, j_m\} = \{1, \ldots, m\}$*.*

"Proposition" (L.–Vallette 2023)

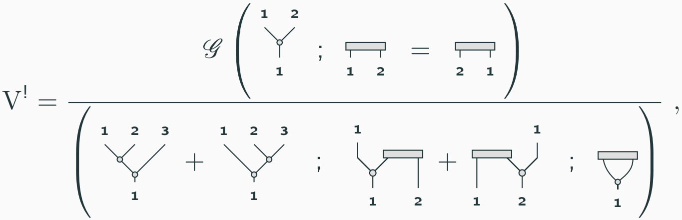
We give an explicit description of the differential ∂_Δ and the structure of DPois_∞-gebra.

The example of V 1

Recall that



where the first generator has degree 0 and the second one has degree -2 .



where the first generator has degree -1 and the second one has degree 1 .

The example of V 3

Question (by Poirier–Tradler)

We do not know if the properad V is Koszul or not.

However, we can consider the $\Omega(\mathrm{V}^{\mathrm{i}})$-gebras, called V_∞-gebras.

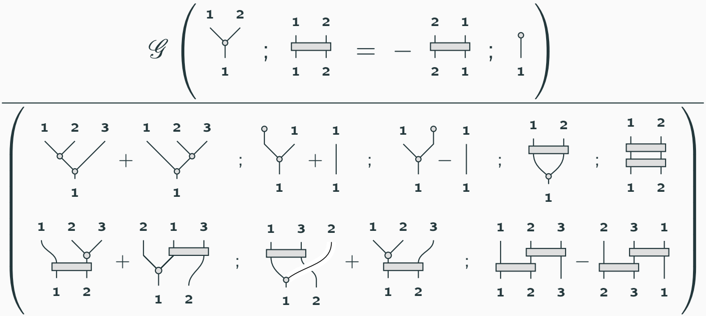
Problem

We do not have a explicit description of $\Omega(\mathrm{V}^{\mathrm{i}})$. (work in progress)

Pre-Calabi–Yau algebras from a properadic point of view

An extension of $\mathrm{DPois}^{!}$

We consider the unital extension of $\mathrm{DPois}^{!}$.

$\mathrm{uDPois}^{!} =$  ,

where $\nu =$  has degree -1 ,  has degree -1 ,

and where $u =$  has degree 1 .

Attention

The linear dual of $\mathrm{uDPois}^{!}$ is **not** a coproperad because the terms with zero input produce infinite sums in the decomposition map.

Definition : codioperad

A codioperad is a coproperad with decomposition map producing only **graphs with genus zero**.

Example

$\mathrm{DPois}^{\text{¡}}$ is a coaug. codioperad, because the relation  in $\mathrm{DPois}^{!}$ kills all graphs with genus > 0 in the decomposition map.

Pre-Calabi–Yau algebras

Example

$\left(\frac{\mathrm{uDPois}^!}{\mathbb{K}u}\right)^*$, where $u =$  is a coaug. codioperad. because

- the relations  and  kill all graphs with genus > 0 in the decomposition map;
- we have killed $u =$  which produced infinite sums in the decomposition map of the linear dual of $\mathrm{uDPois}^!$.

We denote it $\mathrm{C}_{\mathrm{pCY}}$.

Proposition (L.–Vallette 2023)

The $\Omega(\mathrm{C}_{\mathrm{pCY}})$-gebras are pre-Calabi–Yau algebras.

It is a corollary of a result of Yeung and Kontsevich–Takeda–Vlassopoulos : they show that pre-Calabi–Yau algebras are encoded by a cobar of codioperad of multi-corollas.

Produce codioperad from coproperad

Proposition (L.–Vallette 2023)

Let $P = \mathscr{G}(E)/\langle R\rangle$ be a properad with $E(1,0) = 0$. Then $P_{\mathrm{g}=0} := \mathscr{G}_{\mathrm{g}=0}(E)/\langle R_{\mathrm{g}=0}\rangle$ defines by graphs of genus 0 and the composition along graphs of genus 0 is a sub-properad of P. Moreover, $(P_{\mathrm{g}=0})^*$ is a codioperad.

Example

We have $\mathrm{V}^!_{\mathrm{g}=0}$ is a sub-properad of $\mathrm{V}^!$. Then we have a surjection

$$\mathrm{V}^{¡} \twoheadrightarrow (\mathrm{V}^!_{\mathrm{g}=0})^*.$$

Theorem (L.–Vallette 2023)

The codioperads $(\mathrm{V}^!_{\mathrm{g}=0})^*$ *and* $\mathrm{C}_{\mathrm{pCY}}$ *are isomorphic.*

Resume

Theorem (L.–Vallette 2023)

We have surjections of coproperads

$$\mathrm{V}^{i} \twoheadrightarrow (\mathrm{V}^{!}_{\mathrm{g}=0})^{*} \cong \mathrm{C}_{\mathrm{pCY}} \twoheadrightarrow \mathrm{DPois}^{i}$$

which induce surjections of properads

$$\mathrm{V}_{\infty} \twoheadrightarrow \Omega\mathrm{C}_{\mathrm{pCY}} \twoheadrightarrow \mathrm{DPois}_{\infty},$$

and so the inclusion functors

$$\mathrm{DPois}_{\infty}\text{-geb} \to \Omega\mathrm{C}_{\mathrm{pCY}}\text{-geb} \to \mathrm{V}_{\infty}\text{-geb}$$

where morphisms are morphisms of dg vector spaces which commute with the structure.

Consequences for pre-Calabi–Yau algebras

The description of pre-Calabi–Yau algebras as gebras over the properad $\Omega \mathrm{C}_{\mathrm{pCY}}$ implies many consequences.

In the rest of this talk, we will consider $\mathscr{C}$ a coaug. coproperad. You can think that $\mathscr{C}$ is

- $\mathrm{DPois}^{\text{¡}}$ for double Poisson gebras up to homotopy ;
- $\mathrm{C}_{\mathrm{pCY}}$ for pre-Calabi–Yau algebras ;
- $\mathrm{V}^{\text{¡}}$ for V_∞-gebras.

Existence of ∞-morphisms : description of structure

Let $\mathscr{C}$ be a coaug. coproperad and let A be a dg-vector space. Recall that a $\Omega\mathscr{C}$-gebra structure on A is a morphism of (dg) properads $\Omega\mathscr{C} \to \mathsf{End}_A$.

Proposition (Vallette, Hoffbeck–L.–Vallette 2020)

A morphism of properads $\Omega\mathscr{C} \to \mathsf{End}_A$ corresponds to a map $\alpha\colon \overline{\mathscr{C}} \to \mathsf{End}_A$ of $\mathfrak{S}$-bimodules of degree -1 satisfying the Maurer–Cartan equation

$$\partial\alpha + \alpha \star \alpha = 0$$

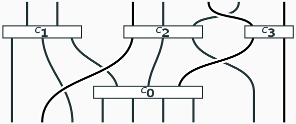
where :

- $\partial\alpha = d_{\mathsf{End}_A} \circ \alpha - \alpha \circ d_{\mathscr{C}}$;
- $\alpha \star \alpha : \mathscr{C} \xrightarrow{\Delta_{(1,1)}} \mathscr{C} \boxtimes_{(1,1)} \mathscr{C} \xrightarrow{\alpha\boxtimes\alpha} \mathsf{End}_A \boxtimes \mathsf{End}_A \xrightarrow{\text{compo.}} \mathsf{End}_A$.

(Remind that the isomorphism $\mathrm{MC}(\mathsf{Hom}(\mathscr{C}, \mathsf{End}_A)) \cong \mathsf{Hom}_{\mathrm{operad}}(\Omega\mathscr{C}, \mathsf{End}_A)$ presented by Wai-Kit)

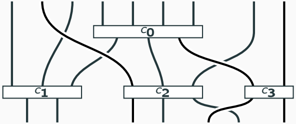
Existence of ∞-morphisms : some notations

- For $\mathscr{C}$ a coproperad, using the counit of $\mathscr{C}$, we have the surjection $\mathscr{C} \boxtimes \mathscr{C} \twoheadrightarrow \mathscr{C} \lhd \mathscr{C}$ where the terms of $\mathscr{C} \lhd \mathscr{C}$ are of the form



with $c_0 \in \overline{\mathscr{C}}$.

- We have also a surjection $\mathscr{C} \boxtimes \mathscr{C} \twoheadrightarrow \mathscr{C} \rhd \mathscr{C}$ where the terms of $\mathscr{C} \rhd \mathscr{C}$ are of the form



with $c_0 \in \overline{\mathscr{C}}$.

- For A and B two dg vector spaces, Hom_B^A the $\mathfrak{S}$-bimodule given, for $(s, e) \in \mathbb{N}^2$ by

$$\mathrm{Hom}_B^A(s, e) = \mathrm{Hom}_{\mathbb{K}}(A^{\otimes e}, B^{\otimes s})$$

Existence of ∞-morphisms : definition

Definition (Hoffbeck–L.–Vallette 2020)

Let $\mathscr{C}$ be a coaug. coproperad. Let $(A, \alpha\colon \overline{\mathscr{C}} \to \mathrm{End}_A)$ and $(B, \beta\colon \overline{\mathscr{C}} \to \mathrm{End}_B)$ be two $\Omega\mathscr{C}$-gebras.

An ∞-morphism $\varphi\colon A \rightsquigarrow B$ is a morphism $\varphi\colon \mathscr{C} \to \mathrm{Hom}_B^A$ of $\mathfrak{S}$-bimodules satisfying the Maurer–Cartan equation

$$\partial\varphi = \varphi \rhd \alpha - \beta \lhd \varphi.$$

where

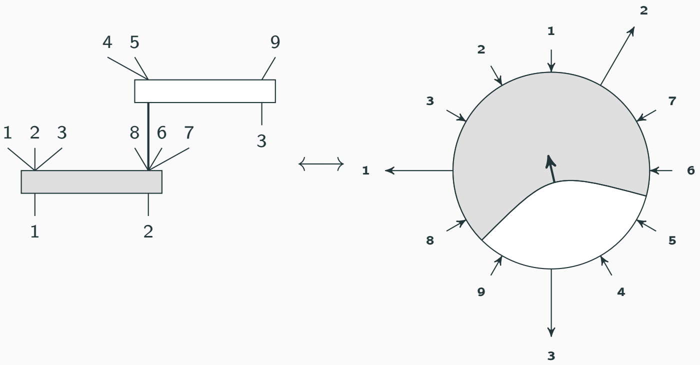
- $\varphi \rhd \alpha\colon \mathscr{C} \xrightarrow{\Delta} \mathscr{C} \boxtimes \mathscr{C} \twoheadrightarrow \mathscr{C} \rhd \mathscr{C} \xrightarrow{\varphi \rhd \alpha} \mathrm{Hom}_B^A \boxtimes \mathrm{End}_A \xrightarrow{\text{compo.}} \mathrm{Hom}_B^A$
- $\beta \lhd \varphi\colon \mathscr{C} \xrightarrow{\Delta} \mathscr{C} \boxtimes \mathscr{C} \twoheadrightarrow \mathscr{C} \lhd \mathscr{C} \xrightarrow{\beta \lhd \varphi} \mathrm{End}_B \boxtimes \mathrm{Hom}_B^A \xrightarrow{\text{compo.}} \mathrm{Hom}_B^A$

We denote by φ_0 the composition $I \xrightarrow{\eta} \mathscr{C} \xrightarrow{\varphi} \mathrm{Hom}_B^A$, where η is the coaug. of $\mathscr{C}$ which corresponds to a chain map $\varphi_0\colon A \to B$.

Existence of ∞-morphisms : recognize an other definition

Proposition (L.–Vallette 2023)

For pre-Calabi–Yau algebras, this definition coincides with the definition of morphisms given by Kontsevich–Takeda–Vlassopoulos.

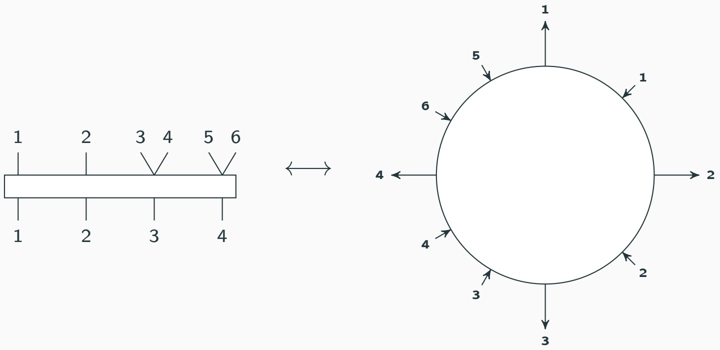


(On the right, you can recognize the splitting presented by Wai-Kit this morning)

Combinatoric for the decomposition map

Important remark

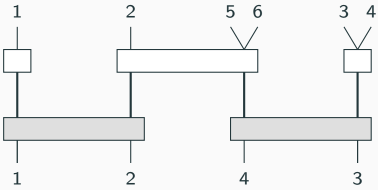
The combinatoric of "roundabouts" is useful to describe the decomposition map of $\mathrm{DPois}^{\mathsf{i}}$ or $\mathrm{C}_{\mathrm{pCY}}$.



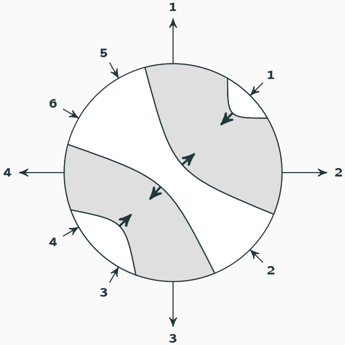
Combinatoric for the decomposition map

Important remark

The combinatoric of "roundabouts" is useful to describe the decomposition map of $\mathrm{DPois}^{\mathsf{i}}$ or $\mathrm{C}_{\mathrm{pCY}}$.



$\longleftrightarrow$



Existence of ∞-morphisms : composition

Definition–Proposition (Hoffbeck–L.–Vallette 2020)

Let $\varphi : (A, \alpha) \rightsquigarrow (B, \beta)$ and $\psi : (B, \beta) \rightsquigarrow (C, \gamma)$ two ∞-morphisms of $\Omega\mathscr{C}$-gebras. The composite of ∞-morphisms $\psi \circledcirc \varphi$ is defined by

$$\psi \circledcirc \varphi : \mathscr{C} \xrightarrow{\Delta} \mathscr{C} \boxtimes \mathscr{C} \xrightarrow{\psi \boxtimes \varphi} \mathrm{Hom}_C^B \boxtimes \mathrm{Hom}_B^A \xrightarrow{\text{compo.}} \mathrm{Hom}_C^A .$$

The category $\infty - \Omega\mathscr{C}$-geb is the category of $\Omega\mathscr{C}$-gebras with ∞-morphisms and there is a canonical functor $\Omega\mathscr{C}\text{-geb} \to \infty - \Omega\mathscr{C}\text{-geb}$.

Corollary (L.–Vallette 2023)

We have the commutative diagram of inclusion functors

$$\begin{array}{ccccc}
\mathrm{DPois}_\infty\text{-geb} & \longrightarrow & \Omega\mathrm{C}_{\mathrm{pCY}}\text{-geb} & \longrightarrow & \mathrm{V}_\infty\text{-geb} \\
\downarrow & & \downarrow & & \downarrow \\
\infty - \mathrm{DPois}_\infty\text{-geb} & \longrightarrow & \infty - \Omega\mathrm{C}_{\mathrm{pCY}}\text{-geb} & \longrightarrow & \infty - \mathrm{V}_\infty\text{-geb}
\end{array}$$

Good properties of ∞-morphisms

Let $\varphi : (A, \alpha) \rightsquigarrow (B, \beta)$ be an ∞-morphism.

- φ is an ∞-isomorphism if $\varphi_0 : A \to B$ is an isomorphism.

Proposition (Hoffbeck–L.–Vallette 2020)

φ is invertible in $\infty - \Omega\mathscr{C}$-geb if and only if φ is an ∞-isomorphism.

- φ is an ∞-quasi-isomorphism if $\varphi_0 : A \to B$ is a quasi-isomorphism.

Proposition (Hoffbeck–L.–Vallette 2020)

If φ is an ∞-quasi-isomorphism, then there exists an ∞-quasi-isomorphism $\psi : (B, \beta) \rightsquigarrow (A, \alpha)$ whose ψ_0 induces the homology inverse of φ_0.

Transfer theorem

Recall that a *contraction* of a dg vector space (A, d_A) is another dg vector space (H, d_H) equipped with chain maps i and p and a homotopy h of degree 1

$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H) \quad ,$$

satisfying

$$pi = \mathrm{id}_H \ , \quad \mathrm{id}_A - ip = d_A h + h d_A \ , \quad hi = 0 \ , \quad ph = 0 \ , \quad \text{and} \quad h^2 = 0 \ .$$

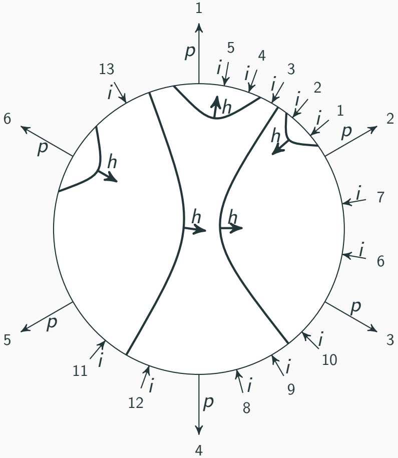
Theorem (Hoffbeck–L.–Vallette, 2020)

Let (A, α) be a $\Omega\mathscr{C}$-gebra and let H be a contraction of A. There exists an (explicit) structure of $\Omega\mathscr{C}$-gebra on H and (explicit) extensions of i and p into ∞-quasi-isomorphisms.

Remark about transfer for pre-Calabi–Yau structures

Proposition (L.–Vallette 2023)

The transferred structure of a pre-Calabi–Yau algebra is explicitly given in terms of the underlying combinatoric"



A first step to understand $\Omega\mathscr{C}\text{-geb}[(\xrightarrow{\sim})^{-1}]$

Theorem (Hoffbeck–L.–Vallette to appear in 2024)

Two $\Omega\mathscr{C}$-gebras are ∞-quasi-isomorphic if and only if they are related by a zig-zag of (strict) quasi-isomorphisms:

$$\begin{array}{c} \exists\ \infty\text{-quasi-isomorphism} \\ (A,\alpha) \stackrel{\sim}{\rightsquigarrow} (B,\beta) \end{array} \iff \begin{array}{c} \exists\ \text{zig-zag of quasi-isomorphisms} \\ (A,\alpha) \xrightarrow{\sim} \bullet \xleftarrow{\sim} \bullet \cdots\cdots \bullet \xleftarrow{\sim} \bullet \xrightarrow{\sim} (B,\beta) \end{array} .$$

This result will be useful to prove some formality results.

Next step

Hoffbeck–L.–Vallette to appear in 2024

We construct a simplicial enrichment $\Delta - \Omega\mathscr{C}$-geb of the category $\Omega\mathscr{C}$-geb such that

$$\pi_0(\Delta - \Omega\mathscr{C}\text{-geb}) \cong \mathrm{ho}(\infty - \Omega\mathscr{C}\text{-geb}).$$

Thanks for your attention.