Inverting smooth Calabi-Yau structures in practice

Pre-Calabi-Yau day, Grenoble

Alex Takeda

Uppsala University

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KTV1 = Kontsevich-T-Vlassopoulos, arXiv:2112.14667
KTV2 = Kontsevich-T-Vlassopoulos, arXiv:2301.01567
RTW = Rivera-T-Wang, arXiv:2308.09684

Introduction

Smooth CY and pre-CY

Earlier this morning, we heard about the algebraic definition of a pre-CY structure, and about the relation between smooth CY/pre-CY structures on some algebra/category to (derived) symplectic geometry on its space of representations.

I would like to start with two complementary perspectives on this relation.

Partially defined oriented 2d TQFTs

A CY algebra A is the data of an oriented 2d TQFTs which assigns $HH_*(A)$ to the circle.

- Costello (2006): explicit description for proper CY structures on dg categories. Recall:

$$\omega : HH_*(A) \to \Bbbk[-n], \text{factoring through } HH_*(A) \to HC_*(A) \cong HH_*(A)_{S^1}$$

  is proper CY structure if induces $\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(Y, X)^\vee[-n]$. Gives TQFT structure:

$$\mathbf{C}_*^{\text{cell}}(\mathscr{M}_{g,r,s}^{\text{framed}}, \mathcal{L}^d) \otimes C_*(A)^{\otimes r} \to C_*(A)^{\otimes s} \tag{1}$$

  for all $r \geq 1, s \geq 0$.That is, 2d cobordisms with at least one input, or gen. by handles of index 1 and 2
- Lurie (2010): abstractly, dual story should exist for smooth CY structures, that is: 1 for all $r \geq 0, s \geq 1$, 2d cobordisms with at least one output, or gen. by handles of index 0 and 1
- KTV1: explicit description of partially-defined TQFT structure on $HH_*(A)$ for (A, m) an A_∞-category with n-dim. pre-CY structure: 1 for all $r \geq 1, s \geq 1$, 2d cobordisms with at least one input and at least one output, or gen. by handles of index 1 only

Noncommutative geometry

'Kontsevich-Soibelman' perspective on A_∞-algebras:

$$A_\infty\text{-algebra}(A,\mu) = \text{noncommutative space } X_A + \text{homological vector field } Q$$

Have a correspondence between Hochschild cochains of A and vector fields on X_A.
Picking basis x_i, as a graded vector space, A = ring of functions on X_A, , if $\mu^r(x_{i_1},\ldots,x_{i_r}) = \sum c_{i;i_1,\ldots,i_r} x_i$, then $Q = \sum c_{i;i_1,\ldots,i_r} x_{i_1}\ldots x_{i_r}\partial_i$. Homological if $[\mu,\mu]=0 \Leftrightarrow [Q,Q]=0$.

Can extend this correspondence

- negative cyclic homology of A and closed differential forms on X_A, $d_{dR} \leftrightarrow B$.
- 'higher' Hochschild cohomology of A and polyvector fields on X_A, necklace bracket $\leftrightarrow$ Schouten-Nijenhuis bracket.

Under (1), smooth CY structure on $A \leftrightarrow$ symplectic form on X_A. Under (2), pre-CY structure on $A \leftrightarrow$ Poisson bivector field on X_A.

The two perspectives suggest that, on a smooth A, one should be able to construct a pre-CY structure from a smooth CY structure. NC geometry perspective suggests that there is a notion of nondegeneracy for pre-CY structures = nondegeneracy of Poisson bivector in differential geometry. Also suggest that smooth CY and nondegenerate pre-CY should be in an inverse relation.

Let us write $m = \mu + m_{(2)} + m_{(3)} + \dots, m_{(k)}$ an element of cohomological degree $nk - n - 2k + 4$ in

$$C^*_{(k)}(A) = \prod \mathrm{Hom}(A[1]^{\otimes(r_1+\dots+r_k)}, A^{\otimes k})$$

Proposition 1

When A is smooth, there is a quasi-isomorphism $C^*_{(k)}(A) \simeq \mathrm{Hom}_{A^e}(A, (A^!)^{\otimes_A (k-1)})$.

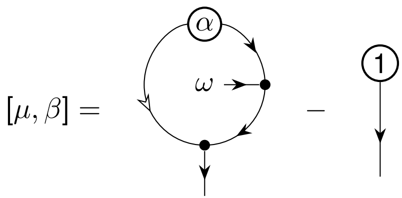
Definition 1

(Kontsevich-Vlassopoulos ca. 2013) The pre-CY structure m is nondegenerate if $\alpha = m_{(2)}$ corresponds to a quasi-isomorphism of A-bimodules $A \xrightarrow{\sim} A^![d]$.

For the other direction, have quasi-isomorphism $C_*(A) \simeq \mathrm{Hom}_{A^e}(A^!, A)$. This gives an equivalent definition:

Definition 2

The pre-CY structure m is nondegenerate if there exists $\omega \in C_*(A)$ and $\beta \in C^*(A)$ such that



These definitions are equivalent (KTV2); the second one can be seen as a chain-level version of the first, and says that α and ω are inverse to each other.

Introduction

Nondegeneracy condition

Not every α can be lifted to a pre-CY structure. Similarly, not every $\omega \in C_*(A)$ can be lifted to some $\widetilde{\omega} \in CC_*^-(A)$.

Theorem 1

(Kontsevich-Vlassopoulos ca. 2013, Pridham 2015, Yeung 2018, KTV2) Assuming nondegeneracy and up to homological equivalence, these two lifting problems have the same answer. Moreover, there is a 'one-to-one' correspondence between smooth CY structures and nondegenerate pre-CY structures on A.

Note: even if α can be lifted to pre-CY structure, may not be true for every representative of $[\alpha]$. Helps to understand correspondence at chain-level, which is the point of KTV2. We use the perspective that this relation is a noncommutative Legendre transform.

NC Legendre transform
Khudaverdian-Voronov's odd Legendrian transform

Khudaverdian-Voronov (2008) explain how the relationship between a symplectic form and the inverse Poisson bivector field is a type of Legendre transform on odd co/tangent bundles. And generalize to a one-to-one correspondence between 'homotopy symplectic forms' and 'homotopy Poisson brackets'.

$$\{\omega = \omega_2 + \omega_3 + \dots\} \leftrightarrow \{P = P_2 + P_3 + \dots\}$$

where $\omega_2 \in \Omega^2(M)$ is nondegenerate, ω_i closed, P_i is i-polyvector field and P satisfies $[P, P]_{SN} = 0$.

Interpreting ω as a function on the odd tangent bundle ΠTM and P as a function on the odd cotangent bundle $\Pi T_* M$, the relation above is analogous to the fiberwise Legendre transform between functions on total space of a vector bundle and its dual.

Can deform to allow for $P = P_1 + P_2 + P_3 + \dots$, and on the LHS deform the closed condition to $(d - \mathcal{L}_{P_1})\omega = 0$.

NC Legendre transform

Khudaverdian-Voronov's odd Legendrian transform

Summarizing this Legendre transform, from P to ω: pick basis $\{x^i\}$, write $\partial_i = \partial/\partial x^i$ and polyvector field $P \in \mathcal{O}(\Pi T^* M)$ as

$$P = \frac{1}{2!} P_2^{ij} \partial_i \partial_j + \frac{1}{3!} P_3^{ijk} \partial_i \partial_j \partial_k + \dots$$

1. Write the fiberwise derivative $FP : \mathcal{O}(\Pi TM) \to \mathcal{O}(\Pi T^* M)$, given in coordinates by

$$dx^i = \frac{\partial P}{\partial(\partial_i)}$$

   If P_2 nondegenerate, then this is an isomorphism of function rings.
2. Take energy function $e_P = \partial_i \frac{\partial P}{\partial \partial_i} - P = \sum (k-1) \frac{1/k!}{P} {}_k^{i_1 \dots i_k}$
3. Take the inverse image of the energy function $\omega = FP^{-1}(e_P) \in \mathcal{O}(\Pi TM) = \Omega(M)$.

In this case, end up with $\omega = \omega_2 + \omega_3 + \dots$, and relation

$$d_{dR}\, \omega = 0 \Leftrightarrow [P, P]_{SN} = 0$$

NC Legendre transform

Noncommutative version

The description in KTV2 is a noncommutative analog of Khudaverdian-Voronov's Legendre transform. The fiberwise derivative FP is more naturally thought of as a variational derivative. In nc world, would be the tangent to the 'map of moduli spaces'

$$\mathscr{M}(\text{smooth CY structures}) \to \mathscr{M}(\text{pre-CY structures})$$

We know the tangent spaces are $CC_-^*(A)$ on the LHS, and

$$C_{[n]}^*(A) = \bigoplus_{k \geq 1} C_{(k,n)}^*(A)$$

with differential given by $[m, -]$, necklace bracket with a pre-CY structure.

Proposition 2

(KTV2) There is a natural map $\Gamma(m, -) : CC_-^*(A) \to C_{[n]}^*(A)[n+2]$, defined from moduli spaces of 'tubes', which is a quasi-isomorphism if $m_{(2)}$ is nondegenerate.

We then take the 'energy function' $e_m = \sum_{k \geq 2}(k-1)m_{(k)}$.The Legendre transform is

$$m \mapsto (\Gamma(m, -))^{-1}(e_m)$$

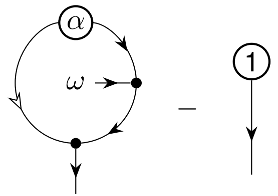
and maps nondegenerate pre-CY structures to smooth CY structures.

Inverting smooth CY structures

In theory

In KTV2, the inverse to $m \mapsto \omega$ is calculated iteratively; we fix ω and start with a symmetrized solution $m_{(2)} = \alpha$ to

$$[\mu, \beta] =$$



then at each stage, prove that MC equation $2[\mu, m_{(\ell)}] = \sum_{i=2}^{\ell-1} [m_{(i)}, m_{(\ell-i+1)}]$ has a solution of the form

$$m_{(\ell)} = \frac{1}{\ell - 1} \Big([\mu, \beta_{(\ell)}] + [m_{(2)}, \beta_{(\ell-1)}] + \cdots + [m_{(\ell-1)}, \beta_{(2)}] + \Gamma^0_{(\ell)}(\omega) + \cdots + \Gamma^{\ell-2}_{(\ell)}(\omega_{\ell-2}) \Big)$$

for some element $\beta_{(2)} + \beta_{(3)} + \cdots + \beta_{(\ell)} \in C^*_{[d]}(A)$ which is guaranteed to exist, and linear combinations of tube quivers $\Gamma_{(\ell)} = \Gamma^0_{(\ell)} + \cdots + \Gamma^{\ell-2}_{(\ell)}$.

Very non-explicit: β depends on 'internal structure' of A, and $\Gamma_{(\ell)}$ is complicated.

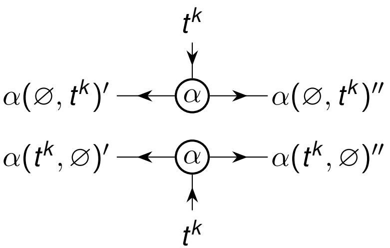
Inverting smooth CY structures

In practice

In practice, it is often enough to calculate α, and then solve for $m_{(3)}, m_{(4)}, \ldots$ using pre-CY equations. Not always guaranteed to work for any A and choice of chain-level α, but in some special cases yes.
Unique chain-level representative: by KTV2, given nondegenerate $[\alpha]$, if its inverse admits a lift to negative cyclic homology, there is some chain representative which admits a lift to a pre-CY structure. If α is uniquely determined from $[\alpha]$, it must be the right class!

Example 1

(RTW) Homology algebra $A = H_*(\Omega S^n, \Bbbk) = \Bbbk[t]$*, homological* $\deg(t) = n - 1 \geq 1$*, with smooth n-dim CY structure given by negative cyclic lift of* $\omega = 1[t]$*. Only chain (up to scaling) in that degree, and similarly the only solution to* α *is*



$$\alpha(\varnothing, t^k)' \otimes \alpha(\varnothing, t^k)'' = \sum_{0 \leq i \leq k-1} t^i \otimes t^{k-1-i}$$

$$\alpha(t^k, \varnothing)' \otimes \alpha(t^k, \varnothing)'' = (-1)^n \sum_{0 \leq i \leq k-1} t^i \otimes t^{k-1-i}$$

In fact, taking $m = \mu + \alpha$ *gives a pre-CY structure.*

Inverting smooth CY structures

In practice

Degree reasons (simpler): sometimes, for degree reasons only finitely number of

$$m_{(k)}^{r_1,\dots,r_k} \in \mathrm{Hom}(A^{\otimes \sum r_i}, A^{\otimes k}), \deg = nk - n - 2k + 4 - \sum r_i$$

can be nonzero.

Example 2

(RTW) Let $A = H_*(\Omega S^1, \mathbb{k}) = \mathbb{k}[t, t^{-1}]$, $\deg(t) = 0$. Want pre-CY structure of dimension 1. Only nonzero possibilities are $(k, (r_i)) = (1, (2)), (2, (0, 1)), (2, (1, 0)), (3, (0, 0, 0))$. Fixing $\omega = t^{-1}[t]$, we have unique solution to inverse problem

$$\alpha(t^k, \varnothing) = \chi_{1 \le k} \left(\frac{1}{2}(1 \otimes t^k + t^k \otimes 1) - \sum_{1 \le i \le k-1} t^i \otimes t^{k-i} \right) + \chi_{k \le -1} \left(\frac{1}{2}(1 \otimes t^k + t^k \otimes 1) + \sum_{k+1 \le i \le -1} t^i \otimes t^{k-i} \right)$$

$\alpha(\varnothing, t^k) = -\alpha(t^k, \varnothing)$ and $m_{(3)} = \frac{1}{4} 1 \otimes 1 \otimes 1$.

Degree reasons (fancier): sometimes can find model with some manifest locality property for bimodules. Recall that for smooth A, we have $C^*_{(k)}(A) \simeq \mathrm{Hom}_{A^e}(A_\Delta, (A^!)^{\otimes_A (k-1)})$.

Example 3

(KTV1) Let X be a Fano variety with $A = \mathrm{End}(E)$ for E generator of $D^b(\mathrm{Coh}(X))$. Have equivalence of categories between A-bimodules and quasi-coherent sheaves on $X \times X$.

$$A_\Delta \leftrightarrow \Delta_* \mathcal{O}_\Delta, \quad A^! \leftrightarrow \Delta_*(\omega_X^{-1})[-n]$$

Therefore can produce α from anticanonical section $\mathcal{O}_X \to \omega_X^{-1}$. Equation for $m_{(k)}$ is

$$[\mu, m_{(k)}] = \textit{closed element of degree } nk - n - 2k + 5 \textit{ in } C^*_{(k)}(A),$$

whose class lives in $\mathrm{Ext}^{-2k+5}(\Delta_*\mathcal{O}, \Delta_*\omega_X^{1-k}) = 0$ when $k \geq 3$ since the two objects are representable by ordinary coherent sheaves in degree zero.

In other words, found local description of the Hom spaces which makes solution to lifting problem manifest.

Poincaré duality

Simplicial path categories

What follows is work in progress with M. Rivera and Z. Wang. We describe how to use the triangulation Λ (simplicial set) of a manifold to put a pre-CY structure on a dg category P_Λ. This simplicial path category has

- Objects: elements of Λ_0, that is, vertices
- Morphisms: 'open necklaces of simplices', with 1-simplices inverted.

This dg category is smooth, and there is an quasi-isomorphism $P_\Lambda \simeq C_*(\Omega|\Lambda|)$.

Poincaré duality

Local Poincaré structures

If the triangulation is fine enough, we define a notion of local Poincaré duality structure on simplicial sets, and prove:

Theorem 2

Given an n-dimensional local Poincaré duality structure on Λ, there is a local chain-level $m_{(2)} \in C^n_{(2)}(P_\Lambda)$.

Locality is a bit complicated, basically says that if necklaces 'above' and 'below' live far away from each other in M, $m_{(2)}$ vanishes. There is also a version if Λ has nontrivial boundary, get a chain-level $m_{(2)}$ which is degenerate. Either way,

Proposition 3

Any local closed element of $C^n_{(2)}(P_\Lambda)$ can be lifted to a pre-CY structure.

Follows from studying vanishing under certain length filtrations, but conceptually from the fact that a small neighborhood of $M \underset{\Delta}{\hookrightarrow} M \times M$ homotopy retracts onto M.