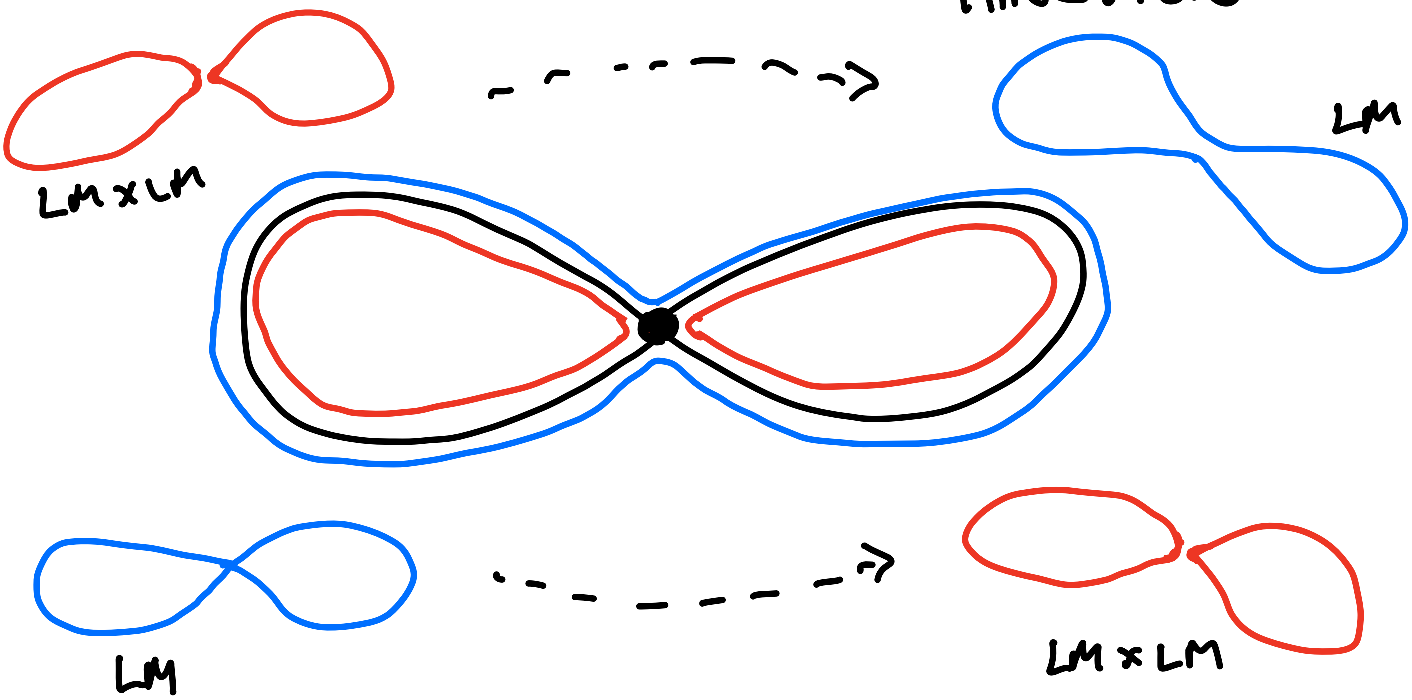
LIFTING THE INTERSECTION PRODUCT ALONG FIBRATIONS: A TOPOLOGICAL POINT OF VIEW ON STRING TOPOLOGY

jt NAEF-RIVERA
HINGSTON



M CLOSED MANIFOLD. $LM = \mathrm{Maps}(S^1, M)$

[JONES] $HH_*(C^*M, C^*M) \cong H^*(LM)$

M 1-CONN
FIELD COEFF

[GOODWILLIE] $HH_*(C_*\Omega M, C_*\Omega M) \cong H_*(LM)$

IDEA: POINCARÉ DUALITY $\longleftrightarrow$ CY STRUCTURE ON C^*M

$\longleftrightarrow$ PRE-CY STRUCTURE ON $C_*\Omega M$

$\rightsquigarrow$ OPERATIONS ON $HH_*(\underline{C^*M}, \underline{C^*M}) / HH_*(\underline{C_*\Omega M}, \underline{C_*\Omega M})$

$\uparrow$ PRODUCT, COPRODUCT $\rightsquigarrow$ PROPERADS
$\rightsquigarrow$ TQFTS ...

INTERSECTION PRODUCT

$$\begin{tikzcd}
H_*(M)\otimes H_*(M) \arrow[r, "AW"] \arrow[d, "PD"', "\cong"] & H_*(M\times M) \arrow[r, dashed] \arrow[d, "\cong"', "PD"] & H_{*-n}(M) \arrow[d, "\cong"', "PD"] \\
H^{*-n}(M)\otimes H^{*-n}(M) \arrow[r] \arrow[rr, bend right, "\cup"'] & H^{*-2n}(M\times M) \arrow[r, "\Delta^*"] & H^{*-2n}(M)
\end{tikzcd}$$

INTERSECTS CYCLES REPRESENTED BY TRANSVERSE SUBMANIFOLD IN M

EQUIVALENTLY: INTERSECTS $A\times B$ WITH ΔM IN $M\times M$

FACT: $f: M_1 \to M_2$ deg 1 MAP

$\Rightarrow$ $f_*: H_*(M_1) \to H_*(M_2)$ RESPECTS THE INTERSECTION PRODUCT.

PROOF: f_* RESPECTS PD & AND $\cup$-PRODUCT

QUESTION 1: $\mathcal{E}$ FIBRATION $\rightsquigarrow$

$$\begin{tikzcd}
\mathcal{E} \arrow[d] \\
M\times M
\end{tikzcd}
\qquad
\begin{tikzcd}
H_*(\mathcal{E}) \arrow[r, dashed, "\exists ?"] \arrow[d] & H_{*-n}(\mathcal{E}|_M) \arrow[d] \\
H_*(M\times M) \arrow[r, "int"] & H_{*-n}(M)
\end{tikzcd}$$

EX: $\mathcal{E} = LM\times LM \xrightarrow{ev_0\times ev_0} M\times M$ $\qquad$ $\mathcal{E}|_M = LM \times_M LM$

$\mathcal{E} = LM \xrightarrow{ev_{0,\frac{1}{2}}} M\times M$ $\qquad$ $\mathcal{E}|_M = \{\gamma(0)=\gamma(\tfrac{1}{2})\}$

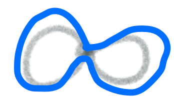
ANSWER: YES!

QUESTION 2: WHAT ARE THE INVARIANCE PROPERTIES?

ANSWER: MORE SUBTLE! WHITEHEAD TORSION APPEARS ...

EXAMPLE: THE CHAS-SULLIVAN PRODUCT

$$\begin{tikzcd} \mathcal{E} = LM \times LM = \mathrm{Maps}(S^1 \sqcup S^1, M) \arrow[r, dashed, "\text{in } H_*"'] \arrow[d, "ev_0 \times ev_0"] & \mathcal{E}|_M = LM \times_M LM \arrow[r, "\text{concat}"] \arrow[d] & LM \\ M \times M \arrow[r, dashed, "\text{in } H_*"'] & M & \end{tikzcd}$$

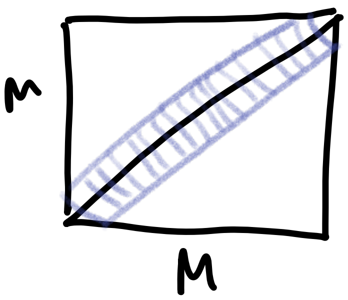


MORE GENERALLY:

$$\begin{tikzcd} \mathcal{E} = \mathrm{Maps}(K_1 \sqcup K_2, M) \arrow[d, "ev_{*_1} \times ev_{*_2}"] \\ M \times M \end{tikzcd} \quad \text{OR} \quad \begin{tikzcd} \mathcal{E} = \mathrm{Maps}(K, M) \arrow[d, "ev_{*_1, *_2}"] \\ M \times M \end{tikzcd}$$

THOM-PONTRJAGIN CONSTRUCTION

ALTERNATIVE DEF OF INTERSECTION PRODUCT



TUBULAR NEIGHBORHOOD U_M

THOM CLASS $\tau_M \in C^n(M\times M, M^c)$

$$\boxed{(A\times B)\cap \Delta M}$$

$$C_*(M\times M) \longrightarrow C_*(M\times M, M^c) \xleftarrow{\sim} C_*(U_M, M^c) \xrightarrow{\boxed{\tau_M\cap}} C_{*-n}(U_M) \underset{\longrightarrow}{\xleftarrow{\sim}} C_{*-n}(M)$$

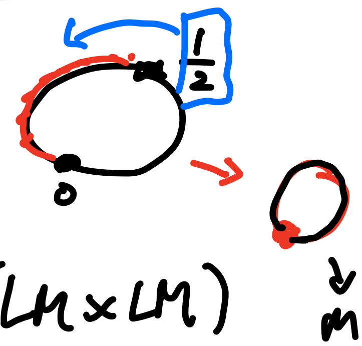
FOR $\begin{array}{c}\mathcal{E}\\ \downarrow p\\ M\times M\end{array}$ $\quad \boxed{p^*\tau_M} \in C^n(\mathcal{E}, \mathcal{E}|_M^c)$

$$C_*(\mathcal{E}) \longrightarrow C_*(\mathcal{E}, \mathcal{E}|_{M^c}) \xleftarrow{\sim} C_*(\mathcal{E}|_{U_M}, \mathcal{E}|_{M^c}) \xrightarrow{\boxed{p^*\tau_M\cap}} C_{*-n}(\mathcal{E}|_{U_M}) \underset{\longrightarrow}{\xleftarrow{\sim}}^{\text{FIB}} C_{*-n}(\mathcal{E}|_M)$$

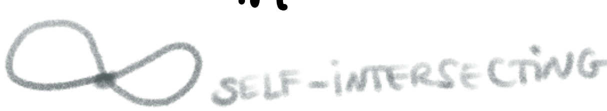
$$= C_*(\mathcal{E}, \mathcal{E}|_M^c)$$

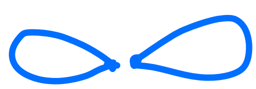
REM:
1) LIFTS THE INTERSECTION PRODUCT
2) MAKES SENSE FOR $\mathcal{E}\longrightarrow U_M$ INSTEAD OF $\mathcal{E}\to M\times M$
3) CAN DEFINE A RELATIVE VERSION.

EX: STRING COPRODUCT: $\quad \begin{array}{c}\mathcal{E} = LM\\ \downarrow ev_{0,\frac12}\\ M\times M\end{array}$



$$\rightsquigarrow H_*(LM) \xrightarrow{int} H_{*-n}(LM|_M) \xrightarrow{cut} H_{*-n}(LM\times LM)$$





[TAMANOI] THIS COPRODUCT IS ESSENTIALLY TRIVIAL!

[GORESKY-HINGSTON] NON-TRIVIAL RELATIVE TO HALF-CONSTANT LOOPS!

$$H_*(LM) \xrightarrow{\times I} H_*(LM\times I, LM\times\partial I) \dashrightarrow H_{*+1-n}(LM\times I|_M, LM\times\partial I)$$

$H_*(LM) \longrightarrow H_{*+1}(LM \times I, \underline{LM \times \partial I})$ ... $\underset{ev_{0,t}}{\longrightarrow} H_{*+1}[LM, \underline{\text{HALF-CONST}}] \dashrightarrow$ ...

THM [NAEF]+[NAEF-RIVERA-W] THE RELATIVE COPRODUCT DISTINGUISHES 3-dim LENS SPACES

$$\mathcal{L}_{p,q} = S^3 \big/ (z_1, z_2) \sim (e^{\frac{2\pi i}{p}} z_1, e^{\frac{2\pi i q}{p}} z_2)$$

$S^3 \subseteq \mathbb{C}^2$

$f : \mathcal{L}_{p,q} \xrightarrow{\simeq} \mathcal{L}_{p,q'}$ HTPY EQUIVALENCE. THEN

$f \simeq$ HOMEO $\iff$ $Lf_* : H_*(L\mathcal{L}_{p,q}, R) \longrightarrow H_*(L\mathcal{L}_{p,q'}, R')$

HALF-CONSTANT LOOPS

RESPECTS THE RELATIVE COPRODUCT.

[NAEF-SAFRONOV] $f : M_1 \xrightarrow{\simeq} M_2$ HOMOTOPY EQUIV.
THE FAILURE OF $Lf_* : H_*(LM_1, R_1) \longrightarrow H_*(LM_2, R_2)$ TO PRESERVE THE COPRODUCT IS MEASURED BY THE TRACE OF WHITEHEAD TORSION.

CONTRAST [COHEN-KLEIN-SULLIVAN] + E
Lf_* PRESERVES THE STRING PRODUCT, ALSO RELATIVE TO "HALF-CONSTANT LOOPS" $= M \times LM \cup LM \times M$

DIFFERENCE?

$$\begin{tikzcd} LM \arrow[d, "ev_{0,\frac{1}{2}}"'] & R = \text{HALF-CST} \arrow[l, hook'] \arrow[d] \\ M \times M & M \arrow[l, "\Delta"'] \end{tikzcd}$$

$$\begin{tikzcd} LM \times LM \arrow[d, "ev_0^2"'] & M \times LM \cup LM \times M \arrow[l, hook'] \arrow[d] \\ M \times M \arrow[r, equal] & M \times M \end{tikzcd}$$

Thm $f: M_1 \to M_2$ HOMOTOPY EQUIVALENCE

(*) $F: (\mathcal{E}_1, R_1) \to (\mathcal{E}_2, R_2)$ MAP OF PAIRS OF FIBRATIONS OVER $f \times f: M_1 \times M_1 \to M_2 \times M_2$. THEN $F_*: H_*(\mathcal{E}_1, R_1) \to H_*(\mathcal{E}_2, R_2)$ RESPECTS THE INT. PRODUCT

(*) IF $R_i \to M_i$ ONLY, THERE IS AN OBSTRUCTION TO INVARIANCE IN $H_*(\mathcal{E}_2, R_2)$.

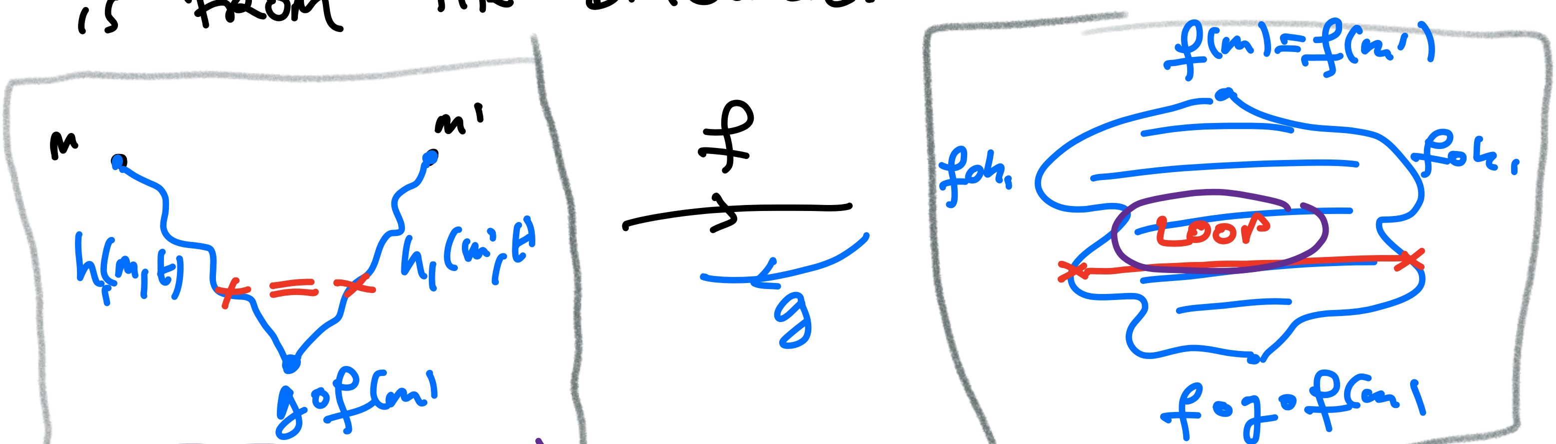
$i = 1, 2$ $\mathcal{E}_i = LM_i \xrightarrow{ev_{0,\frac{1}{2}}} M_i \times M_i$ R_i = HALF-CST

FOR THE STRING COPRODUCT, THE OBSTRUCTION LIVES IN $\boxed{H_1(LM_2, M_2)}$ AND MEASURES HOW "BAD" A HOMOTOPY EQUIVALENCE IS / HOW FAR

$$\boxed{\Delta'_f = \{(m, m') \in M_1 \times M_1 \mid f(m) = f(m')\}}$$

FAKE DIAGONAL

IS FROM THE DIAGONAL.



$\Delta'_f \times I \cap \Delta_1$ $n+1-n=1$

[NAEF-SAFRONOV] ⇒ (?) THIS OBSTRUCTION IS THE DENNIS TRACE OF WHITEHEAD TORSION.