LIFTING THE INTERSECTION PRODUCT ALONG FIBRATIONS: A TOPOLOGICAL POINT OF VIEW ON STRING TOPOLOGY

\[ \mathcal{M} \text{ closed manifold. } \mathcal{LM} = \text{Maps}(S', \mathcal{M}) \]

\[ [\text{Jones}] \quad \mathcal{HH}_*(\mathcal{C}_*^\mathcal{M}, \mathcal{C}_*^\mathcal{M}) \cong \mathcal{H}_*(\mathcal{LM}) \]

\[ \mathcal{M} \text{ 1-connected } \]

\[ \text{Field coeff} \]

\[ [\text{Goodwillie}] \quad \mathcal{H}_*(\mathcal{C}_*^\mathcal{S}\mathcal{M}, \mathcal{C}_*^\mathcal{S}\mathcal{M}) \cong \mathcal{H}_*(\mathcal{LM}) \]

\[ \text{Idea: Poincaré duality } \leftrightarrow \text{ CY structure on } \mathcal{C}_*^\mathcal{M} \]

\[ \leftrightarrow \text{ Pre-CY structure on } \mathcal{C}_*^\mathcal{S}\mathcal{M} \]

\[ \mapsto \text{ Operations on } \mathcal{HH}_*(\mathcal{C}_*^\mathcal{M}, \mathcal{C}_*^\mathcal{M})/\mathcal{H}_*(\mathcal{C}_*^\mathcal{S}\mathcal{M}, \mathcal{C}_*^\mathcal{S}\mathcal{M}) \]

\[ \text{ Product, co-product } \leftrightarrow \text{ Operations } \mapsto \text{ Pre-geometry } \]
**INTERSECTION PRODUCT**

\[
H_*(M) \otimes H_*(M) \xrightarrow{\text{AW}} H_*(M \times M) \xrightarrow{\Delta} H_{*-n}(M)
\]

**FACT:** \( f : H_1 \rightarrow M_2 \) deg 1 map

\( \Rightarrow \) \( f_* : H_*(M_1) \rightarrow H_*(M_2) \) RESPECTS THE INTERSECTION PRODUCT.

**PROOF:** \( f_* \) RESPECTS PD & \( \Omega \) AND U-PRODUCT

**QUESTION 1:** \( E \) Fibration

\[
H_*(E) \rightarrow H_{*-n}(E/M)
\]

**Example:** \( E = L \times \Omega \times L \)

\[
E = L \times \Omega \xrightarrow{ev_0 \times ev_0} M \times M
\]

\( E/M = L \times \Omega L \)

\( E/M = \left[ y(0) = y(1/2) \right] \)
**Question 2:** What are the invariance properties?

**Answer:** More subtle! Whitehead torsion appears...

**Example:** The Chas-Sullivan product

$$E = LM \times LM = \text{Maps}(S'\cup S', M) \xrightarrow{\text{in } H_\ast} E|_M = LM \times LM \xrightarrow{\text{concat}} LM$$

$$\downarrow e_\ast \times e_\ast$$

$$\text{Maps}_{\ast\ast}$$

$$M \times M \xrightarrow{\text{in } H_{\ast\ast}} M$$

**More Generally:**

$$E = \text{Maps}(K_i L K_2, M)$$

$$\downarrow e_{\ast\ast} \times e_{\ast\ast}$$

OR

$$E = \text{Maps}(K_i M)$$

$$\downarrow e_{\ast\ast} \times e_{\ast\ast}$$

$$M \times M$$
Thom–Pontrjagin construction  

**Alternative Def of Intersection Product**

Thom class $T_m \in C^0(M \times M, M^0)$

$C_*(M \times M) \rightarrow C_*(M \times M, M^0) \leftarrow C_*(U_m, M^0)$

For $\mathbf{E}$

$$P^* T_m \in C^0(\mathbf{E}, \overline{\mathbf{E}} | M)$$

$$C_*(M \times M) \rightarrow C_*(M \times M, M^0) \leftarrow C_*(U_m, M^0)$$

**Remark:**
1. Lifts the intersection product
2. Makes sense for $\mathbf{E} \rightarrow U_m$ instead of $\mathbf{E} \rightarrow M \times M$
3. Can define a relative version.

**Example:** String coproduct: $\mathbf{E} = LM$

$$H_*(LM) \xrightarrow{\text{int}} H_{*-n} (\overline{LM} | M) \xrightarrow{\text{cut}} H_{*-n} (LM \times LM)$$

**Tamanoi:** This coproduct is essentially trivial!

**Goresky–Hingston:** Non-trivial relative to half-constant loops!
\[ L_{p,q} = S^3/(z_1, z_2) \sim (e^{2\pi i} z_1, e^{2\pi i} z_2) \]

\[ S^3 \cong \mathbb{C}^2 \]

\[ 0 : L_{p,q} \rightarrow L_{p,q}', \text{ htpy equivalence. Then} \]

\[ 0 \cong \text{homeo} \iff L_{p,q} : H_* (L_{p,q}, R) \rightarrow H_* (L_{p,q}', R) \]

RESPECTS THE RELATIVE COPRODUCT.

[NAEF-SAFRONOV] \[ f : M_1 \cong M_2, \text{ homotopy equiv.} \]

THE FAILURE OF \[ Lf^* : H_*(LM_1, R) \rightarrow H_*(LM_2, R) \]

TO PRESERVE THE COPRODUCT IS MEASURED BY THE TRACE OF WHITEHEAD TORSION.

CONTRAST [COHEN-KLEIN-SULLIVAN] + E

\[ Lf^* \text{ preserves the string product, also relative to "half-constant loops" = M x LM U LM x M} \]

DIFFERENCE?

\[ \text{LM} \leftarrow R = \text{half-cst} \]

\[ \text{ev}_0 : M \xrightarrow{\Delta} M \xrightarrow{\Delta} M \]

\[ M x M \xrightarrow{\Delta} M x M \]

\[ M x LM \leftarrow M x LM x M \]

\[ \text{ev}_0^2 \downarrow \]

\[ M x M \xrightarrow{\Delta} M x M \]
**Theorem** \( f: M_1 \to M_2 \) homotopy equivalence

(\#) \( F: (E_1, R_1) \to (E_2, R_2) \) map of pairs of fibrations over \( f \times f: M_1 \times M_1 \to M_2 \times M_2 \). Then \( F_*: H_2(E_1, R_1) \to H_2(E_2, R_2) \) respects the int. product.

(\#) If \( R_i \to M_i \) only, there is an obstruction to invariance in \( H_2(E_2, R_2) \).

\[ i = 1, 2 \quad E_i = L_{M_i} \quad R_i = \text{half-cst} \]

For the string coproduct, the obstruction lives in \( H_1(L_{E_2}, E_2) \) and measures how "bad" a homotopy equivalence is / how far

\[ \Delta_f = \{ (m, m') \in M_1 \times M_1 / f(m) = f(m') \} \]

is from the diagonal.

This obstruction is the Dennis trace of Whitehead torsion.

[NAEF-SAPROMOV]