The Goresky-Hingston coproduct via pre-CY structures

joint with M. Rivera and A. Takeda

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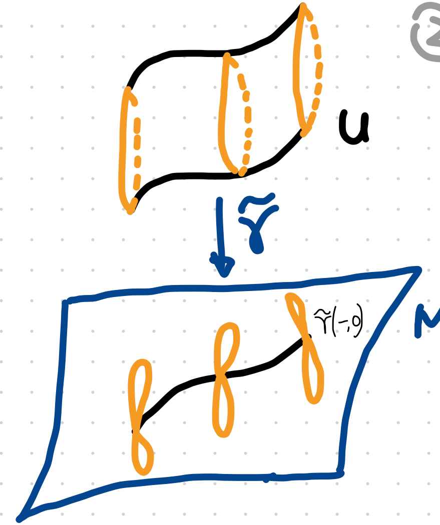
① Motivation



Let M be a manifold of dimension n.

$LM := \mathrm{Map}(S^1, M)$ free loop space

The GH loop coproduct

$$\Delta_{GH}: \underset{\gamma}{H_*(LM,M)} \longrightarrow H_*(LM,M)^{\otimes 2}[n-1]$$

$$\gamma: U \to LM \rightsquigarrow \tilde{\gamma}: U \times S^1 \to M \rightsquigarrow V = \{(u,t) \mid \tilde{\gamma}(u,0) = \tilde{\gamma}(u,t) \;\; t \neq 0\}$$

$$\Delta_{GH}(\gamma): V \longrightarrow LM \times LM$$

$$(u,t) \mapsto$$

$\dim V = \dim U + 1 - n$

This gives $[\Delta_{GH}(r)] \in H_{p+1-n}(LM \times LM, LM \times M \sqcup M \times LM)$.

Δ_{GH} is cocommutative & coassociative & compatible with many interesting operations in string topology.

Question The algebraic analogue of Δ_{GH}?

$$H_*(LM) \cong HH_*(C_*(\Omega M), C_*(\Omega M))$$

Goodwillie

singular chains of the base loop space ΩM

(smooth, Calabi-Yau)

$$\Delta_{GH} \rightsquigarrow \ ?$$

② Chern character for smooth dg algebras

admit bounded projective bimodule resolution

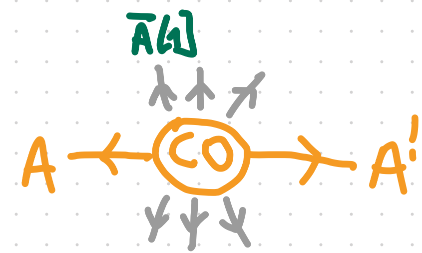
Let A be a smooth dg algebra (i.e. $A \in \mathrm{per}(A \otimes A^{op})$)

Denote $A^! = \mathrm{RHom}_{A^e}(A, A^e)$ the inverse dualising bimodule

Def 1) The coevaluation element co:

$$A \overset{L}{\underset{A^e}{\otimes}} A^! \xrightarrow{\sim} \mathrm{RHom}_{A^e}(A,A)$$

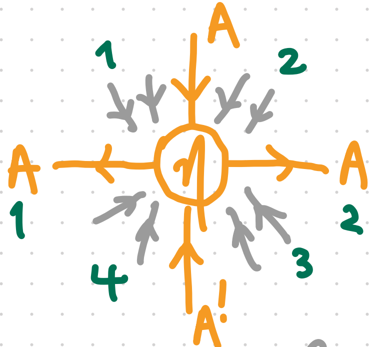
$$co \longmapsto id_A$$



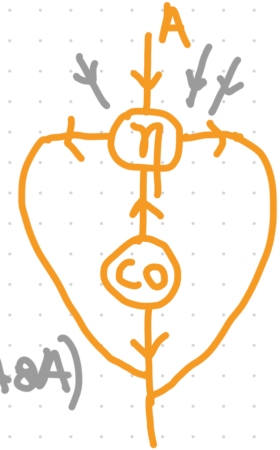
2) The pairing element η:

$$\mathrm{RHom}_{\underset{1234}{A^{\otimes 4}}}(\underline{A \otimes A^!}, \underset{1}{A} \otimes \underset{2}{A}) \xrightarrow{\sim} \mathrm{RHom}_{A^e}(A, \underbrace{\mathrm{RHom}_{A^e}(A^!, A^e)}_{= A^{!!}}) \xrightarrow{\sim} \mathrm{RHom}_{A^e}(A,A)$$

$$\eta \longmapsto id_A$$



$\mapsto$



$\simeq \mathrm{id}_A$

$\in \mathrm{Hom}(\bar{A}[1]^{\otimes p} \otimes A \otimes \bar{A}[1]^{\otimes q}, A)$

$\eta \in \mathrm{Hom}(\bar{A}^{\otimes p} \otimes A \otimes \bar{A}^{\otimes q} \otimes \bar{A}^{\otimes m} \otimes A^! \otimes \bar{A}^{\otimes n}, A \otimes A)$

Def The chern character of A is defined as

$$E_A \in C_*(A,A) \otimes C_*(A,A)$$

Rmk By Polishchuk–Vaintrob,

$[E_A] =$ the one of Shklyarov.

"elbow"

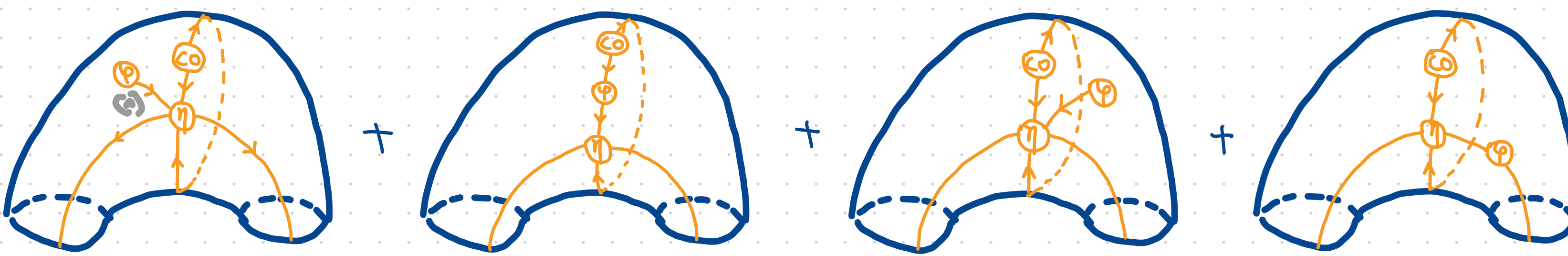
Prop If A admits a non-degenerate pre-CY structure of dim n then $[E_A]$ is $(-1)^n$-symmetric.

i.e. $[E_A] = (-1)^n [E_A^{op}]$

③ The algebraic GH coproduct

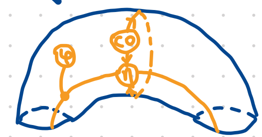
Consider $G: C^*(A\,A) \longrightarrow C_*(A,A)^{\otimes 2}[1]$ defined as

$\varphi \longmapsto$



$E = \sum E_1 \otimes E_2$

Observation $G(d\varphi)+dG(\varphi) = \sum (\varphi\cap E_1)\otimes E_2 \pm \sum E_1\otimes(\varphi\cap E_2)$



If $[E]=0$ i.e. $\exists\, H\in C_*(A,A)^{\otimes 2}$ s.t. $dH=E$ then

$$\begin{aligned}\tilde{\Delta}_H : C^*(A,A) &\longrightarrow C_*(A,A)^{\otimes 2}[-1]\\ \varphi &\longmapsto G(\varphi) \pm (\varphi\cap H')\otimes H'' \pm H'\otimes(\varphi\cap H'')\end{aligned}$$

is a chain map.

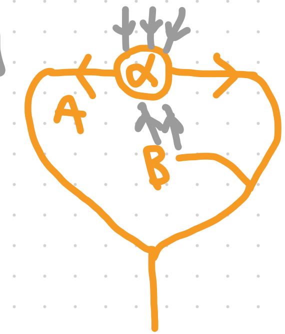
Def A trivialisation of E onto a subcomplex $W\subseteq C_*(A,A)$ is a pair (E_0,H) such that $dH=E-E_0$ where $E_0\in W\otimes C_*(A,A)+C_*(A,A)\otimes W$.

Def Let (E_0, H) be a trivialisation of E onto $W \subseteq \mathcal{C}_*(A\,A)$.

Define $\tilde{\Delta}_H : C^*(A\,A) \longrightarrow \bar{\mathcal{C}}_*(A\,A)^{\otimes 2}[-1]$ $\qquad \bar{\mathcal{C}}_*(A\,A) \triangleq \mathcal{C}_*(A,A)/W$

$R\,\mathrm{Hom}_{A^e}(A, A^!)$ $\qquad \varphi \longmapsto G(\varphi) \pm (\varphi \cap H') \otimes H'' \pm H' \otimes (\varphi \cap H'')$

Let $\alpha \in C^{n}_{(2)}(A\,A)$. Recall $g_\alpha : \mathcal{C}_*(A,A) \longrightarrow C^*(A,A)[n]$.



Define the algebraic GH product

$$\Delta_H : \mathcal{C}_*(A,A) \xrightarrow[\tilde{\Delta}_H \cdot g_\alpha]{} \bar{\mathcal{C}}_*(A\,A)^{\otimes 2}[n-1]$$

Rmk If $[E]=0$ then we may take $W=0$, $E_0=0$ so that

$$\Delta_H : \mathcal{C}_*(A\,A) \longrightarrow \mathcal{C}_*(A,A)^{\otimes 2}$$

$A^i = 0$ for $i<0$

Thm (Rivera–Takeda–W.) Let A be a smooth connective dg algebra, equipped with a nondegenerate pre-CY structure. Suppose (E_0, H) is a trivialisation on a "balanced" $W \subseteq \mathcal{S}_*(A,A)$. If H is appropriately symmetric then $(H_*(\overline{\mathcal{S}}_*(A,A)), \Delta_H)$ is a graded cocommutative and coassociative coalgebra
$(n \geq 3)$

Rmk 1) The coassociativity may fail for $n<3$.

2) For $A = \mathcal{S}_*(\Omega M)$, one option for W is the space spanned by $\chi(M) \cdot id_b$, the constant loop of $b \in M$.

If $\chi(M) = 0$ then $W = 0$.

④ Proof through Efimov's category $\widehat{A_\infty}$

Let A be a dg algebra. Recall that $\widehat{A_\infty}$ is the dg algebra

$$C^*(A, A\otimes A^\vee) \hookrightarrow C^*(A, \mathrm{Hom}_k(A,A)) \twoheadrightarrow \widehat{A_\infty}$$

the quotient

Rmk "Koszul dual" of singularity category by Buchweitz and Orlov.

$D_{sg}(B) = D^b(B)/\mathrm{Perf}\,B$

Prop (RTW) If A carries a nondegenerate smooth pre-CY structure of dim n, then

$M_\alpha = A \oplus A^\vee[1-n]$

1) $\widehat{A_\infty} \simeq M_\alpha$ where $M_\alpha = \mathrm{cone}(f_\alpha : A^\vee[n] \to A)$

2) $\pi_\alpha : C_*(A, M_\alpha)^{\otimes 2} \longrightarrow C_*(A, M_\alpha)[n]$ graded com. & ass. at homology.

$\quad\| $
$C_*(A, A^\vee[1-n]) \oplus C_*(A,A)$

Thm (RTW) Let A be a nondegenerate pre-CY algebra. Suppose that (E_0, H) is a trivialisation of E onto a "balanced" $W \subseteq C_*(A,A)$. Consider $q : C_*(A, A^\vee)[n] \to C_*(A,A)$

$$\begin{tikzcd} C_*(A,A^\vee)[n] \arrow[r, "q"] \arrow[d, "g_\alpha"'] & C_*(A,A) \\ C^*(A,A^\vee) \arrow[ur, "\#E_0"'] & \end{tikzcd}$$

$C^*(A,A^\vee) \otimes C_*(A,A) \xrightarrow{\langle -,- \rangle} k$

$q \sim C_*(A, f_\alpha)$

If H is appropriately symmetric then

1) there is a commutative and associative algebra $(H_*(\mathrm{Ker}\, q), \pi_H)$ $(n \geq 3)$ lifting from π_α on $C_*(A, M_\alpha)$

2) We have the duality $\langle \pi_H(x_1 \otimes x_2), g_\alpha(y) \rangle = \pm \langle x_2 \otimes x_1, (g_\alpha \otimes g_\alpha)\Delta_H(y) \rangle$

$x_1, x_2 \in H_*(\mathrm{Ker}\, q)$ $\quad y \in HH_*(A,A)$

Rmk The balanced condition $\Rightarrow$ $H_*(\mathrm{Ker}\, q) \simeq H_*(\overline{C}_*(A,A))^\vee$.

$\langle \mathrm{Ker}(q), g_\alpha(w) \rangle = 0$

$$
\begin{tikzcd}
\mathrm{Ker}(q) \arrow[d] & \otimes & C_*(A,A) \arrow[d, "g_\alpha"] \arrow[dr, dashed] & \\
C_*(A,A^\vee) & \otimes & C^*(A,A) \arrow[r, "\langle -,- \rangle"] & k
\end{tikzcd}
$$

Example $M=S^{2N}$ $\quad N\geq 2$ $\quad C_*(\Omega M)\simeq k[t]$ $\quad |t|=2N-1$ $\quad \operatorname{char} k\neq 2$

$$HH_i(A,A)=\begin{cases} k\,t^{2k-1} & \text{for } i=|t|(2k-1) \quad k>0\\ k\,t^{2k}[t] & \text{for } i=|t|(2k+1)+1 \quad k\geq 0\end{cases}$$

The chern character $E=2\cdot(1\otimes 1)\in C_0(A,A)^{\otimes 2}$.

Take the trivialisation (E_0,H) onto $W=k\cdot 1\subseteq A\subseteq C_0(A,A)$
and $E_0=E$, $H=0$

then $\Delta_H : HH_*(A,A)/W \to HH_*(A,A)/W^{\otimes 2}$

$$t^{2k+1}\longmapsto -\sum_{i=1}^{k} t^{2i-1}\otimes t^{2k+1-2i}$$

$$t^{2k}[t]\longmapsto -\sum_{i=1}^{k} t^{2i-1}\otimes t^{2k-2i}[t]+\sum_{i=0}^{k-1} t^{2i}[t]\otimes t^{2k-2i}$$

Thank you !