A disk means:

- The 2-dimensional disk with punctures on boundary
- Each punctured is assigned either: 
  
  "+" ("input") or "−" ("output").

We only remember the diffeomorphism type of a disk.
Fix $A \in \text{grVect}_k$.

Given a disk $D$, a $D$-shaped map on $A$ is a $k$-linear map

$$A \otimes \Sigma^+ \rightarrow A \otimes \Sigma^-$$

where $\Sigma^+ = \{"+"\text{ punctures}\}$ and $\Sigma^- = \{"-"\text{ punctures}\}$.

Example.

$$\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a \\
 + \\
 + \\
 c \\
 - \\
 - \\
 b \\
 + \\
 +
\end{array}
\end{array}
\end{array}$$

$$F(a, b, c) \in A \otimes A$$

The ordering is important.
Pre-Calabi-Yau algebras

We will consider a collection of maps $\pi$ which to each disk $D$ assigns a $D$-shaped map on $A$

$$\pi(D) : A \otimes \Sigma^+ \rightarrow A \otimes \Sigma^-$$

**Definition**

The graded vector space $A$, together with this collection of maps $\pi$ is said to be a pre-Calabi-Yau algebra if it satisfies

$$\pi \circ \pi = 0$$
Gluing of disks

We can glue disks along punctures with opposite polarity.

We can also compose the maps as we glue:

This gives a $D$-shaped map $A^\otimes 3 \rightarrow A^\otimes 4$. 
The condition $\pi \circ \pi = 0$

Suppose we are given a collection $\pi$ of maps. For any given disk $D$,

Each way of writing $D$ as a gluing $D = D_1 \# D_2$

$\implies$ a $D$-shaped map $\pi(D_1) \circ \pi(D_2)$

The condition $\pi \circ \pi = 0$ then says:

For each disk $D$, we require

$$\sum_{D = D_1 \# D_2} \pm \pi(D_1) \circ \pi(D_2) = 0$$
The condition $\pi \circ \pi = 0$

i.e., we require that, for each disk $D$, we have

$$\sum_{D_1 \cup D_2} \pm \pi \pi = 0$$
Example 1

Consider the collection that is nonzero only on the shape

\[
\begin{array}{c}
\mu \\
\downarrow \\
A
\end{array}
\]

There are two ways to combine two of this disk:

\[
\begin{align*}
\mu(\mu(f, g), h) - \mu(f, \mu(g, h)) &= 0 \\
\end{align*}
\]

Thus \((A, \mu)\) is a (non-unital) associative algebra.
Example 2

Consider the collection that is nonzero only on the shapes

\[ \sum_{\mu(r+s+t+1)} \pm \mu_r \mu_t + \mu_s(a_{r+1}, \ldots, a_{r+s}), a_{r+s+1}, \ldots, a_n) = 0 \]

i.e., \((A, \mu_1, \mu_2, \ldots)\) is an \(A_\infty\)-algebra.
Example 3) Consider the collection that is nonzero only on the shape

$$\Delta : A \rightarrow A \otimes A$$

Then \((A, \Delta)\) is a coassociative coalgebra.

Example 4) Consider the collection that is nonzero only on the shapes

$$\mu : A \otimes A \rightarrow A \quad \text{and} \quad \Delta : A \rightarrow A \otimes A$$

Then \((A, \mu, \Delta)\) is an infinitesimal bialgebra that satisfies a certain derivation property.
Example 5

Consider the collection that is nonzero only on the shapes

We will rewrite $P(a, b) = \{\{a, b\}\} \in A \otimes A$.

0) The disk on the right has an internal $C_2$ symmetry. Accordingly, $\{\{-, -\}\} : A \otimes A \to A \otimes A$ is required to be $C_2$-invariant.

Gluing these disks gives rise to three kinds of disks:

1) First disk $\Rightarrow (A, \mu)$ is an associative algebra.
2) Second disk $\Rightarrow \{\{-, -\}\}$ is a derivation in each variable.
3) Third disk $\Rightarrow \{\{-, -\}\}$ satisfies the “double Jacobi identity”.

i.e., $(A, \mu, P)$ is a double Poisson algebra.
A closer look

Fix $A \in \text{grVect}_k$. Fix $m \in \mathbb{Z}$. Define

$$\mathcal{X}^{(p)}(A; m) = \left\{ \begin{aligned}
\text{Collection } F \text{ that assigns a } D\text{-shaped map } \\
F(D) : (A[1]) \otimes \Sigma^- \to (A[-m]) \otimes^p \\
to each disk } D \text{ with } p \text{ outputs}
\end{aligned} \right\}$$

Theorem [Kontsevich-Vlassopoulos]

There is a graded Lie bracket

$$\{-,-\} : \mathcal{X}^{(p)}(A; m) \otimes \mathcal{X}^{(q)}(A; m) \to \mathcal{X}^{(p+q-1)}(A; m)$$

given by the diagram
This bracket gives a graded Lie algebra structure on

\[ \hat{\mathfrak{X}}^{\geq 1}(A; m)[m + 1] := \prod_{p \geq 1} \mathfrak{X}^{(p)}(A; m)[m + 1] \]

**Definition**

Let \( m = 2 - n \). An \( n \)-pre-Calabi-Yau algebra is a graded vector space \( A \), together with a Maurer-Cartan element in the graded Lie algebra \( \hat{\mathfrak{X}}^{\geq 1}(A; m)[m + 1] \).

In other words, we have \( \pi = \pi_1 + \pi_2 + \pi_3 + \ldots \) satisfying \( \{ \pi, \pi \} = 0 \). From now on, we ignore the homological shifts, and so we neglect \( m \).
Recall that the bracket \([-,-]\) has weight grading \(-1\):
\[
\{-,-\} : \mathfrak{x}^{(p)}(A) \otimes \mathfrak{x}^{(q)}(A) \to \mathfrak{x}^{(p+q-1)}(A)
\]
In particular, \(\mathfrak{x}^{(1)}(A)\) is a Lie subalgebra and \(\mathfrak{x}^{(>1)}(A)\) is a Lie ideal.

Write \(\pi = \pi_1 + \pi_{\geq 2}\).

Then the condition \(\{\pi, \pi\} = 0\) splits into two conditions
1) \(\{\pi_1, \pi_1\} = 0\)
2) \(\{\pi_1, \pi_{\geq 2}\} + \frac{1}{2}\{\pi_{\geq 2}, \pi_{\geq 2}\} = 0\)

Thus, a pre-Calabi-Yau algebra is always an \(A_\infty\)-algebra \((A, \pi_1)\) with extra structure \(\pi_{\geq 2}\).
pre-Calabi-Yau structures

\[ \{-, -\} : \mathcal{X}^p(A) \otimes \mathcal{X}^q(A) \to \mathcal{X}^{p+q-1}(A) \]

Given an \( A_\infty \) structure \( \pi_1 \), then \( \{\pi_1, -\} \) preserves each component \( \mathcal{X}^q(A) \), so that it becomes a chain complex.

**Definition**

The graded vector space \( \mathcal{X}^p(A) \) together with the differential 

\( d_{\pi_1} = \{\pi_1, -\} \)

is called the cyclic invariant higher Hochschild cochains on the \( A_\infty \) algebra \( (A, \pi_1) \).

Thus, \( (\mathcal{X}^\bullet(A), d_{\pi_1}) \) becomes a DG Lie algebra.

**Definition**

A pre-Calabi-Yau structure on the \( A_\infty \) algebra \( (A, \pi_1) \) is a Maurer-Cartan element in the DG Lie algebra \( (\hat{\mathcal{X}}_{\geq 2}(A), d_{\pi_1}) \).
Recall the definition

\[ \mathcal{X}^{(p)}(A) = \left\{ \begin{array}{l}
\text{Collection } F \text{ that assigns a } D\text{-shaped map } \\
F(D) : A \otimes \Sigma^{-} \rightarrow A \otimes^p \\
to each disk } D \text{ with } p \text{ outputs}
\end{array} \right\} \]

A disk with \( p \) outputs is completely determined by the number of inputs between the consecutive outputs:

For example, this disk is specified by the sequence \((3, 2, 0, 2, 1)\). Any cyclic rotation, e.g., \((2, 0, 2, 1, 3)\), defines the same disk.
Higher Hochschild cochains

Thus we have

$$\mathcal{X}^{(p)}(A) = \left\{ \begin{array}{l} \text{Collection } F \text{ that assigns a } D\text{-shaped map} \\
F(D) : A \otimes \Sigma^p \to A \otimes^p \\
to each disk } D \text{ with } p \text{ outputs} \end{array} \right\}$$

$$= \left[ \prod_{(n_1, \ldots, n_p) \in \mathbb{N}^p} \text{Hom}_k(A \otimes^{n_1} \ldots \otimes A \otimes^{n_p}, A \otimes^p) \right] C_p$$

$$= \left[ R\text{Hom}_{(A \otimes^p)e}(A \otimes^p, \tau(A \otimes^p)_{id}) \right] C_p$$

1) $\text{Hom}_{Be}(-, -)$ means $B$-bimodule map.

2) Recall that $A$ has a free $A$-bimodule resolution

$$\ldots \to A \otimes A \otimes^2 \otimes A \to A \otimes A \otimes^1 \otimes A \to A \otimes A \otimes^0 \otimes A \to A \to 0$$

Accordingly, $\text{Bar}(A) \otimes^p$ is a $A \otimes^p$-bimodule resolution of itself.
$X$ is a smooth manifold (or variety).
Then a Poisson structure is a bivector field $\pi_2 \in \mathfrak{X}^2(X)$ satisfying
$\{\pi_2, \pi_2\} = 0$.

In other contexts (e.g., deformation quantization, derived algebraic geometry, etc), it is natural to consider generalized Poisson structures

$$\pi_{\geq 2} = \pi_2 + \pi_3 + \ldots$$

satisfying the Maurer-Cartan equation.

pre-Calabi-Yau structures is a noncommutative analogue of Poisson structures.
### Noncommutative calculus

<table>
<thead>
<tr>
<th>Commutative</th>
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<tbody>
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**Wai-kit Yeung**  
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Introduction to pre-Calabi-Yau structures  
November 3, 2023  
19 / 32
**Definition**

A **symplectic structure** on $X$ is a **closed 2-form** whose **underlying 2-form** determines an isomorphism

$$\Omega^1(X)^\vee \cong \Omega^1(X)$$

of sheaves.

**Definition [Ginzburg, Kontsevich-Vlassopoulos, Brav-Dyckerhoff]**

An **$n$-Calabi-Yau structure** on $A$ is a **negative cyclic homology class** $\tilde{\eta} \in HC_n^-(A)$ whose **underlying Hochschild homology class** $\eta \in HH_n(A)$ determines an isomorphism

$$A^\vee[n] \cong A$$

in the derived category of DG bimodules.
Two flavors of noncommutative algebraic geometry

NC1
For any notion $P$ on varieties/derived stacks, etc, its **noncommutative generalization** should be a notion $\tilde{P}$ on associative algebras, such that it reduces to $P$ for (smooth) commutative algebras.

NC2 [The Kontsevich-Rosenberg principle]
For any structure $P$ on varieties/derived stacks, etc, its **noncommutative analogue** should be a structure $P_{nc}$ on an associative algebra $A$ which induces the structure $P$ on the moduli space of representations of $A$.

For example, a Calabi-Yau structure is a noncommutative generalization of a Calabi-Yau variety; and a noncommutative analogue of a symplectic structure.
### The Kontsevich-Rosenberg principle

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All these can be justified by the Kontsevich-Rosenberg principle.
Actually, the way I see it, there are two sides of NC2:

- **Phenomenological side:**
  - The Kontsevich-Rosenberg principle

- **Ontological side:**
  - Use the analogy

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together with some aesthetic principles, guided by some basic examples, to develop noncommutative geometry.

Keypoint: These two sides end up doing the same thing!
Symplectic and Poisson structure on moduli spaces

Theorem [Pantev-Toën-Vaquié-Vezzosi, Brav-Dyckerhoff, Y.]
Any $n$-Calabi-Yau structure on $A$ induces a $(2 - n)$-shifted symplectic structure on the derived moduli stack of representations of $A$.

Theorem [Y.]
Any $n$-pre-Calabi-Yau structure on $A$ induces a $(2 - n)$-shifted Poisson structure on the derived moduli stack of representations of $A$. 
## Koszul duality principle

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<th>Cuntz-Quillen resolution</th>
<th>Bar resolution</th>
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<tr>
<td>Extended NC differential forms</td>
<td>Cyclic bar complex</td>
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<tr>
<td>Smooth/left Calabi-Yau structures</td>
<td>Compact/right Calabi-Yau structure</td>
</tr>
<tr>
<td>Extended necklace Lie algebra</td>
<td>Higher Hochschild cochains</td>
</tr>
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<th>$\mathcal{D}_{coh}^b(X)$</th>
<th>$\mathcal{D}_{perf}(X)$</th>
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<tr>
<td>WFuk($X$)</td>
<td>Fuk($X$)</td>
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</table>

One can often use formulas on one side to guess the formula on the other side.
Ribbon dioperads

Classical picture:

Modules ← Algebras ← Operads

Proposed picture:

Modules ← Algebras ← Operads

... ← Dioperads

... ← Properads ← regular patterns

... ← pre-CY algebras ← Ribbon dioperads
The theory of operads, revisited

Recall that an operad consists of:

1. an $\mathbb{S}$-module $\mathcal{P}$.
2. a collection of composition maps on $\mathcal{P}$
3. that satisfies certain axioms.

In fact, all the three items flow out of the combinatorics of rooted trees.

e.g., Let $\mathcal{f}$ be the category of rooted corolla (morphisms are isomorphisms of rooted corolla), then an $\mathbb{S}$-module is the same as a functor $\mathcal{P} : \mathcal{f} \to \text{Vect}$.

The theory of regular patterns [Getzler] or Feynman categories [Kauffmann-Ward] allows us to replace the combinatorics of rooted trees by that of other types of graphs.

Replace rooted trees by directed ribbon trees $\Rightarrow$ theory of ribbon dioperads
For each graded vector space $V$, there is an operad $\text{End}(V)$. Then a $\mathcal{P}$-algebra structure on $V$ is a map of operad $\mathcal{P} \to \text{End}(V)$.

If $\mathcal{C}$ is a dg co-operad (all operads and co-operads are assumed to be non-unital), then $\text{Hom}_{\text{S-Mod}}(\mathcal{C}, \mathcal{P})$ is a dg Lie algebra. Moreover, there is a bijection

$$\text{MC}(\text{Hom}_{\text{S-Mod}}(\mathcal{C}, \mathcal{P})) \cong \text{Hom}_{\text{dg-operads}}(\Omega(\mathcal{C}), \mathcal{P})$$
Let $\mathcal{f}$ be the category of directed ribbon corollas, with morphisms being isomorphisms of directed ribbon corollas (i.e., same as the “disks” earlier in this talk).

Then a ribbon dioperad consists of:

1. an $\mathcal{f}$-module. i.e., a functor $\mathcal{P} : \mathcal{f} \rightarrow \text{grVect}_k$.
2. a collection of composition maps on $\mathcal{P}$
3. that satisfies certain axioms.

where the appropriate structure maps (2) and axioms (3) flow out of the combinatorics of directed ribbon trees.
Ribbon dioperads

(1) For any $A \in \text{grVect}_K$, there is a ribbon dioperad $\text{End}(A)$. Namely,

$$\text{End}(A)(D) = \text{Hom}_K(A \otimes \Sigma^+, A \otimes \Sigma^-)$$

(2) There is a ribbon co-dioperad $C$, such that $C(D) = K$ for all directed ribbon corolla $D$. Then we have

$$\text{Hom}_{\text{f-Mod}}(C, \text{End}(A)) \cong \prod_{p \geq 0} \left\{ \prod_{(n_1, \ldots, n_p) \in \mathbb{N}^p} \text{Hom}_K(A \otimes n_1 \otimes \ldots \otimes A \otimes n_p, A \otimes p) \right\}^{C_p}$$
Ribbon dioperads

Theorem [Y.]
For an ribbon co-dioperad $\mathcal{C}$ and any ribbon dioperad $\mathcal{P}$, the graded vector space $\text{Hom}_{f-\text{Mod}}(\mathcal{C}, \mathcal{P})$ has a graded Lie algebra structure.

Theorem [Y.]
There is a cobar construction that gives

$$\text{MC}(\text{Hom}_{f-\text{Mod}}(\mathcal{C}, \mathcal{P})) \cong \text{Hom}_{\text{dg-ribbon-dioperad}}(\Omega(\mathcal{C}), \mathcal{P})$$

In particular, consider $\mathcal{P} = \text{End}(A)$ and $\mathcal{C} = \mathbb{K}$, then $\Omega(\mathbb{K})$ (cobar of a point) is the dg ribbon dioperad that controls pre-Calabi-Yau structures.
For $C = \mathbb{K}$, we have

$$\text{Hom}_{f-\text{Mod}}(C, \text{End}(A))$$

$$\cong \prod_{p \geq 0} \left[ \prod_{(n_1, \ldots, n_p) \in \mathbb{N}^p} \text{Hom}_k(A^\otimes n_1 \otimes \cdots \otimes A^\otimes n_p, A^\otimes p) \right] C_p$$

In positive characteristics, we should not take cyclic invariants.

Problem: resolve $C = \mathbb{K}$ so that it is projective as $f$-modules.

Remark: this resembles the problem of finding an $E_\infty$-operad (i.e., an operad that resolves the constant operad $\mathbb{K}$ so that it is projective as $S$-modules).