# Introduction to pre-Calabi-Yau structures

Wai-kit Yeung
wkyeung@proton.me

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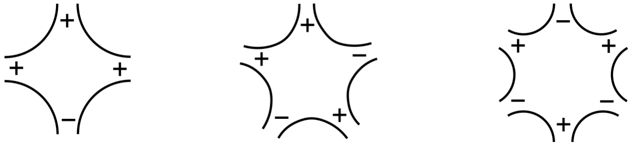
# Disk

A disk means:

- The 2-dimensional disk with punctures on boundary
- Each punctured is assigned either:
  "$+$" ("input") or "$-$" ("output").




We only remember the diffeomorphism type of a disk.

# D-shaped maps

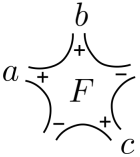
Fix $A \in \mathrm{grVect}_k$.

Given a disk $D$, a $D$-**shaped map on** $A$ is a $k$-linear map

$$A^{\otimes \Sigma^+} \to A^{\otimes \Sigma^-}$$

where $\Sigma^+ = \{\text{"}+\text{" punctures}\}$ and $\Sigma^- = \{\text{"}-\text{" punctures}\}$.

Example.




$$F(a, b, c) \in A \otimes A$$

The ordering is important.

# Pre-Calabi-Yau algebras

We will consider a **collection of maps** $\pi$ which to each disk $D$ assigns a $D$-shaped map on $A$

$$\pi(D) : A^{\otimes \Sigma^+} \to A^{\otimes \Sigma^-}$$

**Definition**

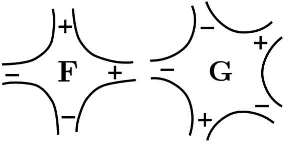
The graded vector space $A$, together with this collection of maps $\pi$ is said to be a pre-Calabi-Yau algebra if it satisfies

$$\pi \circ \pi = 0$$

# Gluing of disks

We can glue disks along punctures with opposite polarity.

We can also compose the maps as we glue:




This gives a $D$-shaped map $A^{\otimes 3} \to A^{\otimes 4}$.

# The condition $\pi \circ \pi = 0$

Suppose we are given a collection $\pi$ of maps.

For any given disk D,

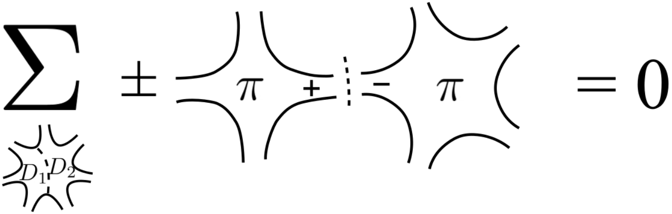
$$\begin{aligned} & \text{Each way of writing } D \text{ as a gluing } D = D_1 \# D_2 \\ \Longrightarrow & \text{ a D-shaped map } \pi(D_1) \circ \pi(D_2) \end{aligned}$$

The condition $\pi \circ \pi = 0$ then says:

For each disk $D$, we require

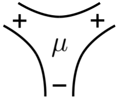
$$\sum_{D = D_1 \# D_2} \pm \pi(D_1) \circ \pi(D_2) = 0$$

i.e., we require that, for each disk $D$, we have



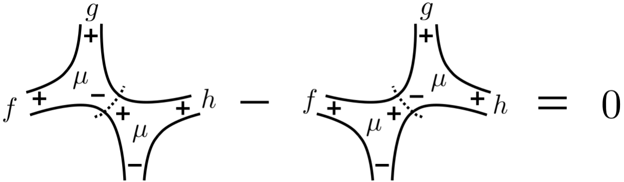
# Example 1

Consider the collection that is nonzero only on the shape




$$\begin{array}{c} A \otimes A \\ \downarrow \mu \\ A \end{array}$$

There are two ways to combine two of this disk:



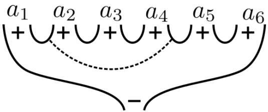

$$\mu(\mu(f, g), h) - \mu(f, \mu(g, h)) = 0$$

Thus $(A, \mu)$ is a (non-unital) associative algebra.

# Example 2

Consider the collection that is nonzero only on the shapes


The requirement $\pi \circ \pi = 0$ gives one relation for each $n \geq 1$:




$$\sum_{r+s+t=n} \pm\mu_{r+t+1}(a_1, \ldots, a_r, \mu_s(a_{r+1}, \ldots, a_{r+s}), a_{r+s+1}, \ldots, a_n) = 0$$

i.e., $(A, \mu_1, \mu_2, \ldots)$ is an $A_\infty$-algebra.

# Example 3, 4

Example 3) Consider the collection that is nonzero only on the shape

$$\Delta : A \to A \otimes A$$

Then $(A, \Delta)$ is a coassociative coalgebra.

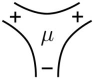
Example 4) Consider the collection that is nonzero only on the shapes

$$\mu : A \otimes A \to A \qquad \text{and} \qquad \Delta : A \to A \otimes A$$

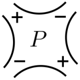
Then $(A, \mu, \Delta)$ is an infinitesimal bialgebra that satisfies a certain derivation property.

# Example 5

Consider the collection that is nonzero only on the shapes




$$\begin{array}{c} A \otimes A \\ \downarrow \mu \\ A \end{array}$$

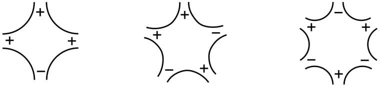



$$\begin{array}{c} A \otimes A \\ \downarrow P \\ A \otimes A \end{array}$$

We will rewrite $P(a, b) = \{\{a, b\}\} \in A \otimes A$.

0) The disk on the right has an internal $C_2$ symmetry. Accordingly, $\{\{-, -\}\} : A \otimes A \to A \otimes A$ is required to be $C_2$-invariant.

Gluing these disks gives rise to three kinds of disks:



1) First disk $\Rightarrow (A, \mu)$ is an associative algebra.
2) Second disk $\Rightarrow \{\{-, -\}\}$ is a derivation in each variable.
3) Third disk $\Rightarrow \{\{-, -\}\}$ satisfies the "double Jacobi identity".

i.e., $(A, \mu, P)$ is a double Poisson algebra.

# A closer look

Fix $A \in \mathrm{grVect}_k$. Fix $m \in \mathbb{Z}$. Define

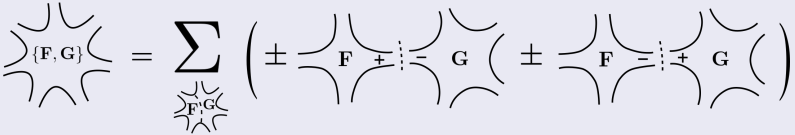
$$\mathfrak{X}^{(p)}(A;m) = \left\{ \begin{array}{l} \text{Collection } F \text{ that assigns a } D\text{-shaped map} \\ F(D) : (A[1])^{\otimes \Sigma^-} \to (A[-m])^{\otimes p} \\ \text{to each disk } D \text{ with } p \text{ outputs} \end{array} \right\}$$

## Theorem [Kontsevich-Vlassopoulos]

There is a graded Lie bracket

$$\{-,-\} : \mathfrak{X}^{(p)}(A;m) \otimes \mathfrak{X}^{(q)}(A;m) \to \mathfrak{X}^{(p+q-1)}(A;m)$$

given by the diagram



# A closer look

This bracket gives a graded Lie algebra structure on

$$\hat{\mathfrak{X}}^{\geq 1}(A; m)[m+1] := \prod_{p \geq 1} \mathfrak{X}^{(p)}(A; m)[m+1]$$

**Definition**

Let $m = 2 - n$. An $n$-pre-Calabi-Yau algebra is a graded vector space $A$, together with a Maurer-Cartan element in the graded Lie algebra $\hat{\mathfrak{X}}^{\geq 1}(A; m)[m+1]$.

In other words, we have $\pi = \pi_1 + \pi_2 + \pi_3 + \ldots$ satisfying $\{\pi, \pi\} = 0$.

From now on, we ignore the homological shifts, and so we neglect $m$.

# pre-Calabi-Yau structures

Recall that the bracket $\{-,-\}$ has weight grading $-1$:

$$\{-,-\} : \mathfrak{X}^{(p)}(A) \otimes \mathfrak{X}^{(q)}(A) \to \mathfrak{X}^{(p+q-1)}(A)$$

In particular, $\mathfrak{X}^{(1)}(A)$ is a Lie subalgebra and $\mathfrak{X}^{(>1)}(A)$ is a Lie ideal.

Write $\pi = \pi_1 + \pi_{\geq 2}$.
Then the condition $\{\pi, \pi\} = 0$ splits into two conditions
1) $\{\pi_1, \pi_1\} = 0$
2) $\{\pi_1, \pi_{\geq 2}\} + \frac{1}{2}\{\pi_{\geq 2}, \pi_{\geq 2}\} = 0$

Thus, a pre-Calabi-Yau algebra is always an $A_\infty$-algebra $(A, \pi_1)$ with extra structure $\pi_{\geq 2}$.

$$\{-,-\} : \mathfrak{X}^{(p)}(A) \otimes \mathfrak{X}^{(q)}(A) \to \mathfrak{X}^{(p+q-1)}(A)$$

Given an $A_\infty$ structure $\pi_1$, then $\{\pi_1, -\}$ preserves each component $\mathfrak{X}^{(q)}(A)$, so that it becomes a chain complex

**Definition**

The graded vector space $\mathfrak{X}^{(p)}(A)$ together with the differential $d_{\pi_1} = \{\pi_1, -\}$ is called the cyclic invariant higher Hochschild cochains on the $A_\infty$ algebra $(A, \pi_1)$.
Thus, $(\mathfrak{X}^\bullet(A), d_{\pi_1})$ becomes a DG Lie algebra.

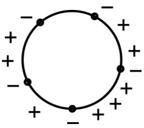
**Definition**

A pre-Calabi-Yau structure on the $A_\infty$ algebra $(A, \pi_1)$ is a Maurer-Cartan element in the DG Lie algebra $(\hat{\mathfrak{X}}^{\geq 2}(A), d_{\pi_1})$.

Recall the definition

$$\mathfrak{X}^{(p)}(A) = \left\{ \begin{array}{l} \text{Collection } F \text{ that assigns a } D\text{-shaped map} \\ F(D) : A^{\otimes \Sigma^-} \to A^{\otimes p} \\ \text{to each disk } D \text{ with } p \text{ outputs} \end{array} \right\}$$

A disk with $p$ outputs is completely determined by the number of inputs between the consecutive outputs:




For example, this disk is specified by the sequence $(3, 2, 0, 2, 1)$.
Any cyclic rotation, e.g., $(2, 0, 2, 1, 3)$, defines the same disk.

# Higher Hochschild cochains

Thus we have

$$\mathfrak{X}^{(p)}(A) = \left\{ \begin{array}{l} \text{Collection } F \text{ that assigns a } D\text{-shaped map} \\ F(D) : A^{\otimes \Sigma^-} \to A^{\otimes p} \\ \text{to each disk } D \text{ with } p \text{ outputs} \end{array} \right\}$$

$$= \left[ \prod_{(n_1,\dots,n_p)\in\mathbb{N}^p} \mathrm{Hom}_k(A^{\otimes n_1} \otimes \dots \otimes A^{\otimes n_p}, A^{\otimes p}) \right]^{C_p}$$

$$= \left[ \boldsymbol{R}\mathrm{Hom}_{(A^{\otimes p})^e}(A^{\otimes p},\ {}_{\tau}(A^{\otimes p})_{\mathrm{id}}) \right]^{C_p}$$

1) $\mathrm{Hom}_{B^e}(-,-)$ means $B$-bimodule map.

2) Recall that $A$ has a free $A$-bimodule resolution

$$\dots \to A \otimes A^{\otimes 2} \otimes A \to A \otimes A^{\otimes 1} \otimes A \to A \otimes A^{\otimes 0} \otimes A \to A \to 0$$

Accordingly, $\mathrm{Bar}(A)^{\otimes p}$ is a $A^{\otimes p}$-bimodule resolution of itself.

# Noncommutative analogue of Poisson structure

$X$ is a smooth manifold (or variety).
Then a Poisson structure is a bivector field $\pi_2 \in \mathfrak{X}^2(X)$ satisfying $\{\pi_2, \pi_2\} = 0$.

In other contexts (e.g., deformation quantization, derived algebraic geometry, etc), it is natural to consider generalized Poisson structures

$$\pi_{\geq 2} = \pi_2 + \pi_3 + \dots$$

satisfying the Maurer-Cartan equation.

pre-Calabi-Yau structures is a noncommutative analogue of Poisson structures.

# Noncommutative calculus

| **Commutative** | **Noncommutative** |
|---|---|
| Commutative algebra | Associative algebra |
| Derived stacks | DG categories |
| Modules or sheaves | Bimodules |
| Differential forms | Hochschild homology |
| Closed forms | Negative cyclic homology |
| de Rham cohomology | Periodic cyclic homology |
| Symplectic structure | Calabi-Yau structure |
| Vector fields | Hochschild cohomology |
| Polyvector fields | Poly-Hochschild cohomology |
| Poisson structure | pre-Calabi-Yau structure |

# Symplectic structure vs Calabi-Yau structures

## Definition

A **symplectic structure** on $X$ is a **closed** 2-**form** whose **underlying** 2-**form** determines an isomorphism

$$\Omega^1(X)^\vee \xrightarrow{\cong} \Omega^1(X)$$

of sheaves.

## Definition [Ginzburg, Kontsevich-Vlassopoulos, Brav-Dyckerhoff]

An $n$-**Calabi-Yau structure** on $A$ is a **negative cyclic homology class** $\tilde{\eta} \in HC_n^-(A)$ whose **underlying Hochschild homology class** $\eta \in HH_n(A)$ determines an isomorphism

$$A^\vee[n] \xrightarrow{\cong} A$$

in the derived category of DG bimodules.

# Two flavors of noncommutative algebraic geometry

## NC1

For any notion $P$ on varieties/derived stacks, etc, its **noncommutative generalization** should be a notion $\widetilde{P}$ on associative algebras, such that it reduces to $P$ for (smooth) commutative algebras.

## NC2 [The Kontsevich-Rosenberg principle]

For any structure $P$ on varieties/derived stacks, etc, its **noncommutative analogue** should be a structure $P_{nc}$ on an associative algebra $A$ which induces the structure $P$ on the moduli space of representations of $A$.

For example, a Calabi-Yau structure is a noncommutative generalization of a Calabi-Yau variety; and a noncommutative analogue of a symplectic structure.

# The Kontsevich-Rosenberg principle

| **Commutative** | **Noncommutative** |
|---|---|
| Commutative algebra | Associative algebra |
| Derived stacks | DG categories |
| Modules or sheaves | Bimodules |
| Differential forms | Hochschild homology |
| Closed forms | Negative cyclic homology |
| de Rham cohomology | Periodic cyclic homology |
| Symplectic structure | Calabi-Yau structure |
| Vector fields | Hochschild cohomology |
| Polyvector fields | Poly-Hochschild cohomology |
| Poisson structure | pre-Calabi-Yau structure |

All these can be justified by the Kontsevich-Rosenberg principle.

# Two sides of NC2

Actually, the way I see it, there are two sides of NC2:

- Phenomenological side:
  The Kontsevich-Rosenberg principle
- Ontological side:
  Use the analogy

| **Commutative** | **Noncommutative** |
|---|---|
| Commutative algebra | Associative algebra |
| Derived stacks | DG categories |
| Modules or sheaves | Bimodules |

  together with some aesthetic principles, guided by some basic examples, to develop noncommutative geometry.

Keypoint: These two sides end up doing the same thing!

# Symplectic and Poisson structure on moduli spaces

**Theorem [Pantev-Toën-Vaquié-Vezzosi, Brav-Dyckerhoff, Y.]**

Any $n$-Calabi-Yau structure on $A$ induces a $(2-n)$-shifted symplectic structure on the derived moduli stack of representations of $A$.

**Theorem [Y.]**

Any $n$-pre-Calabi-Yau structure on $A$ induces a $(2-n)$-shifted Poisson structure on the derived moduli stack of representations of $A$.

# Koszul duality principle

| | |
|---|---|
| Cuntz-Quillen resolution | Bar resolution |
| Extended NC differential forms<br>Smooth/left Calabi-Yau structures | Cyclic bar complex<br>Compact/right Calabi-Yau structure |
| Extended necklace Lie algebra<br>pre-Calabi-Yau structures | Higher Hochschild cochains<br>pre-Calabi-Yau structures |
| $\mathcal{D}^b_{\mathrm{coh}}(X)$<br>$\mathrm{WFuk}(X)$ | $\mathcal{D}_{\mathrm{perf}}(X)$<br>$\mathrm{Fuk}(X)$ |

One can often use formulas on one side to guess the formula on the other side.

# Ribbon dioperads

Classical picture:

$$\text{Modules} \leftsquigarrow \text{Algebras} \leftsquigarrow \text{Operads}$$

Proposed picture:

$$\begin{tikzcd}
\text{Modules} & \text{Algebras} \arrow[l, squiggly] & \text{Operads} \arrow[l, squiggly] & \\
 & \cdots & \text{Dioperads} \arrow[l, squiggly] & \\
 & \cdots & \text{Properads} \arrow[l, squiggly] & \begin{array}{c}\text{regular}\\ \text{patterns}\end{array} \arrow[l, squiggly] \arrow[ull, squiggly] \arrow[uul, squiggly] \arrow[ddl, squiggly] \\
 & & \vdots & \\
\cdots & \begin{array}{c}\text{pre-CY}\\ \text{algebras}\end{array} \arrow[l, squiggly] & \begin{array}{c}\text{Ribbon}\\ \text{dioperads}\end{array} \arrow[l, squiggly] &
\end{tikzcd}$$

## The theory of operads, revisited

Recall that an operad consists of:

(1) an $\mathbb{S}$-module $\mathcal{P}$.
(2) a collection of composition maps on $\mathcal{P}$
(3) that satisfies certain axioms.

In fact, all the three items flow out of the combinatorics of rooted trees.

e.g., Let $\mathfrak{f}$ be the category of rooted corolla (morphisms are isomorphisms of rooted corolla), then an $\mathbb{S}$-module is the same as a functor $\mathcal{P} : \mathfrak{f} \to \mathrm{Vect}$.

The theory of regular patterns [Getzler] or Feynman categories [Kauffmann-Ward] allows us to replace the combinatorics of rooted trees by that of other types of graphs.

Replace rooted trees by directed ribbon trees $\Rightarrow$ theory of ribbon dioperads

# The theory of operads, revisited

For each graded vector space $V$, there is an operad $\mathrm{End}(V)$. Then a $\mathcal{P}$-algebra structure on $V$ is a map of operad $\mathcal{P} \to \mathrm{End}(V)$.

If $\mathcal{C}$ is a dg co-operad (all operads and co-operads are assumed to be non-unital), then $\mathrm{Hom}_{\mathbb{S}-\mathrm{Mod}}(\mathcal{C}, \mathcal{P})$ is a dg Lie algebra. Moreover, there is a bijection

$$\mathrm{MC}(\mathrm{Hom}_{\mathbb{S}-\mathrm{Mod}}(\mathcal{C}, \mathcal{P})) \cong \mathrm{Hom}_{\mathrm{dg-operads}}(\Omega(\mathcal{C}), \mathcal{P})$$

# Ribbon dioperads

Let $\mathfrak{f}$ be the category of directed ribbon corollas, with morphisms being isomorphisms of directed ribbon corollas (i.e., same as the "disks" earlier in this talk).

Then a ribbon dioperad consists of:

(1) an $\mathfrak{f}$-module. i.e., a functor $\mathcal{P} : \mathfrak{f} \to \mathrm{grVect}_k$.

(2) a collection of composition maps on $\mathcal{P}$

(3) that satisfies certain axioms.

where the appropriate structure maps (2) and axioms (3) flow out of the combinatorics of directed ribbon trees.

# Ribbon dioperads

(1) For any $A \in \mathrm{grVect}_{\mathbb{K}}$, there is a ribbon dioperad $\mathrm{End}(A)$. Namely,

$$\mathrm{End}(A)(D) = \mathrm{Hom}_{\mathbb{K}}(A^{\otimes \Sigma^+}, A^{\otimes \Sigma^-})$$

(2) There is a ribbon co-dioperad $\mathcal{C}$, such that $\mathcal{C}(D) = \mathbb{K}$ for all directed ribbon corolla $D$. Then we have

$$\begin{aligned} &\mathrm{Hom}_{\mathfrak{f}-\mathrm{Mod}}(\mathcal{C}, \mathrm{End}(A)) \\ \cong &\prod_{p \geq 0} \left[ \prod_{(n_1,\ldots,n_p) \in \mathbb{N}^p} \mathrm{Hom}_k(A^{\otimes n_1} \otimes \ldots \otimes A^{\otimes n_p}, A^{\otimes p}) \right]^{C_p} \end{aligned}$$

# Ribbon dioperads

**Theorem [Y.]**

For an ribbon co-dioperad $\mathcal{C}$ and any ribbon dioperad $\mathcal{P}$, the graded vector space $\mathrm{Hom}_{\mathfrak{f}-\mathrm{Mod}}(\mathcal{C}, \mathcal{P})$ has a graded Lie algebra structure.

**Theorem [Y.]**

There is a cobar construction that gives

$$\mathrm{MC}(\mathrm{Hom}_{\mathfrak{f}-\mathrm{Mod}}(\mathcal{C}, \mathcal{P})) \cong \mathrm{Hom}_{\mathrm{dg-ribbon-dioperad}}(\Omega(\mathcal{C}), \mathcal{P})$$

In particular, consider $\mathcal{P} = \mathrm{End}(A)$ and $\mathcal{C} = \mathbb{K}$, then $\Omega(\mathbb{K})$ (cobar of a point) is the dg ribbon dioperad that controls pre-Calabi-Yau structures.

For $\mathcal{C} = \mathbb{K}$, we have

$$\begin{aligned}&\mathrm{Hom}_{\mathfrak{f}-\mathrm{Mod}}(\mathcal{C}, \mathrm{End}(A)) \\ \cong &\prod_{p \geq 0} \left[ \prod_{(n_1, \ldots, n_p) \in \mathbb{N}^p} \mathrm{Hom}_k(A^{\otimes n_1} \otimes \ldots \otimes A^{\otimes n_p}, A^{\otimes p}) \right]^{C_p}\end{aligned}$$

In positive characteristics, we should not take cyclic invariants.

Problem: resolve $\mathcal{C} = \mathbb{K}$ so that it is projective as $\mathfrak{f}$-modules.

Remark: this resembles the problem of finding an $\mathbb{E}_\infty$-operad (i.e., an operad that resolves the constant operad $\mathbb{K}$ so that it is projective as $\mathbb{S}$-modules).